Solutions of the Bethe-Salpeter equation for fermion boson system

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Outline

1. Motivations and generalities: BS Amplitude and BS Equation for a fermion-boson bound system $\rightarrow \mathcal{L}_{int} = \lambda_\phi \phi\phi\chi + \lambda_\psi \bar{\psi}\psi\chi$
2. Nakanishi perturbation-theory integral representation (PTIR) and the BS Amplitude
3. The exact projection of the BSE onto the null plane and the PTIR of BSA
4. Eigenvalues and LF distributions in ladder approximations
5. Conclusions & Perspectives

Motivations

- To achieve a fully covariant description for a few-body system, in Minkowski space
- To take properly into account the dynamics, within a field-theoretical framework, obtaining non perturbative outcomes.
- To make feasable numerical calculations

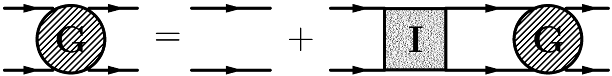
Well-known non perturbative approches: lattice calculations in Euclidean space

The BSE in a nutshell

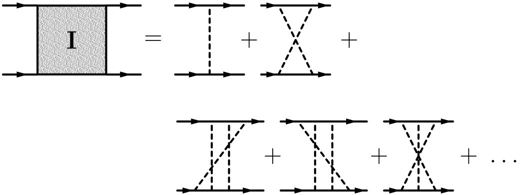
The 4-point Green's Function, schematically is given by

$$G(x_1, x_2; y_1, y_2) = < 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 > \quad ,$$

fulfills an integral equation $G = G_0 \; + \; G_0 \; \mathrm{I} \; G$



I $\equiv$ kernel given by the infinite sum of irreducible Feynmann graphs



Iterations produce all the expected contributions

Insert a complete Fock basis in

$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 >$$

then in the Fourier space, the bound state contribution (assuming only one non degenerate bound state for the sake of simplicity) appears as a pole, i.e.

$$G_B(k, k'; p_B) \simeq \frac{i}{(2\pi)^{-4}} \frac{\phi(k; p_B)\ \bar{\phi}(k'; p_B)}{2\omega_B(p_0 - \omega_B + i\epsilon)}$$

where $\omega_B = \sqrt{M_B^2 + |\mathbf{p}|^2}$ and $\phi(k; p_B)$ is the Bethe-Salpeter Amplitude, in the Fourier space, for a bound state. In configuration space, BS Amplitude is given by

$$\langle 0| T\{\phi_1(x_1)\phi_2(x_2)\}|p_B\, \beta\rangle$$

For $p_0 \to \omega_B$ the 4-point Green's function can be approximate by

$$G \simeq G_B + \ regular\ \ terms$$

and one deduces from $G = G_0 \ + \ G_0 \ \mathrm{I} \ G$, the integral equation determining the BS Amplitude for a bound state, i.e. the homogeneous BS Eq.

$$\psi(k; p_B, \beta) = G_0(k; p_B, \beta) \int \frac{d^4k'}{(2\pi)^4} \, \mathrm{I}(k, k'; p_B) \, \psi(k'; p_B, \beta)$$

with (nor self-energy neither vertex corrections, at the present stage)

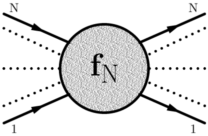
$$G_0 = \frac{i}{(\frac{p_B}{2} - k)^2 - m_\phi^2 + i\epsilon} \ \ \frac{i(\frac{\not p}{2} + \not k + m_\psi^2)}{(\frac{p_B}{2} + k)^2 - m_\psi^2 + i\epsilon}$$

and, in the ladder approximation

$$\mathrm{I}(k, k') = \frac{(-i\lambda_\phi\lambda_\psi)}{(k - k')^2 - \mu^2 + i\epsilon}$$

Feynman parametrization

In the sixties, Nakanishi (PR **130**, 1230 (1963)) proposed an integral representation for N-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.



For N external legs, a generic contribution to the transition amplitude is given by

$$f_{\mathcal{G}}(p_1, p_2, ..., p_N) \propto \prod_{r=1}^{k} \int d^4 q_r \; \frac{1}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2) \; \ldots \; (\ell_n^2 - m_n^2)}$$

where one has n propagators and k loops ($\equiv$ n. of integration variables).
The label $\mathcal{G} \to (n, k)$

Following the standard (textbook) elaboration, one can write

$$f_{\mathcal{G}}(s) \propto \prod_{i=1}^{n} \int_0^1 d\alpha_i \; \frac{\delta(1-\sum_{j=1}^{n}\alpha_j)}{U^2(\alpha)\; [F(n,N,\alpha,s)+i\epsilon]^{n-2k}}$$

where

$$F(n,N,\alpha,s) = -\sum_{j=1}^{n} \alpha_j \; m_j^2 + \sum_{h} \; \eta_h \; s_h$$

with the dependence upon the external momenta, $p_1, p_2 \dots p_N$, traded off in favour of all the independent scalar products $s \equiv \{s_1, s_2, \dots s_h, \dots\}$, one can construct.

Nakanishi PTIR - I

Nakanishi proposal for a compact and elegant espression of the full N-leg amplitude $f_N(s) = \sum_{\mathcal{G}} \; f_{\mathcal{G}}(s)$

Introducing the identity

$$1 \doteq \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\beta}\right) \int_0^\infty d\gamma \; \delta\left(\gamma - \sum_l \frac{\alpha_l m_l^2}{\beta}\right)$$

with $\beta = \sum \eta_i$ and integrating by parts $n - 2k - 1$ times

$$f_{\mathcal{G}}(s) \propto \; \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h) \; \tilde{\phi}_{\mathcal{G}}(z,\gamma)}{(\gamma - \sum_h z_h s_h)}$$

where $\tilde{\phi}_{\mathcal{G}}(z,\gamma)$ is a proper function

The depedence upon the details of the diagram, (n,k), moves from the denominator to the numerator!! The SAME formal expression for the denominator of ANY diagram $\mathcal{G}$ appears

Nakanishi PTIR - II

The full N-leg transition amplitude can be formally written as

$$f_N(s) = \sum_{\mathcal{G}} f_{\mathcal{G}}(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h)\, \phi_N(z,\gamma)}{(\gamma - \sum_h z_h s_h)}$$

where

$$\phi_N(z,\gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(z,\gamma)$$

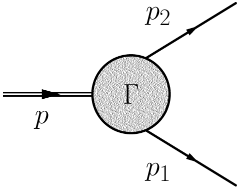
Within the BS framework, such an elegant expression can be exploited for obtaining

- the 3-leg transition amplitude (vertex function $\rightarrow$ bound-state BS amplitude) (Kusaka et al, PRD **56** (1997), Carbonell-Karmanov EPJA **27** (2006))
- the 4-leg one (off-shell or half-off-shell T-matrix $\rightarrow$ scattering-state BS amplitude) (FSV, PRD **85** (2012))

The PTIR of the vertex function

$$f_3(s) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk \cdot p - i\epsilon}$$

with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$



How can the Nakanishi weight function, ϕ_3, be determined for an actual, dynamical model?

Can the Nakanishi expression, elaborated in perturbation theory , be used in a non perturbative realm, as the BS framework does (one has to face with an integral equation, i.e. one has an infinite set of contributions)?

The fermion-boson BSE

In the case of $J^\pi = \frac{1}{2}^+$, the BS amplitude has the following general form

$$\psi(k,p) = \left(\phi_1 \mathbb{1} + \phi_2 \gamma_\mu \frac{k^\mu}{M}\right) u(p,s)$$

Representing each of the BS components $\phi_a(k,p)$ by means of the Nakanishi integral

$$\phi_a(k,p) = \int_0^\infty d\gamma' \int_{-1}^{1} dz' \frac{g_a(\gamma',z')}{\left[k^2 + z' p\cdot k - \kappa^2 - \gamma' + i\epsilon\right]^3}$$

and applying the light-front projection $\int_{-\infty}^{\infty} \frac{dk^-}{2\pi}$ $(k^\pm = k^0 \pm k_z)$

$$\int_0^\infty d\gamma' \frac{g_a(\gamma',z)}{\left[\gamma' + D_0(\gamma,z)\right]^2} = \sum_{a'=1}^{2} \int_0^\infty d\gamma' \int_{-1}^{1} dz' \, \mathcal{V}_{aa'}\left(\gamma,z,\gamma',z'\right) g_{a'}(\gamma',z')$$

with

$D_0(\gamma,z) = \gamma + (1-z^2)\left(\bar{m}^2 - \frac{M^2}{4}\right) + (z\bar{m} + \Delta\bar{m})^2$

Numerical results for eigenvalues and LF distributions in ladder apprx.

We have carried out a comprehensive investigation, in ladder approximation, of the simple fermion-boson model,
$\mathcal{L}_{int} = \lambda_\phi \phi\phi\chi + \lambda_\psi \bar{\psi}\psi\chi$,
varying both binding energies $0 < B/\bar{m} \leq 2$ and the mass of the exchanged scalar, $\mu/\bar{m}$

After fixing the binding energy, $B/\bar{m} = 2 - M/\bar{m}$, and the mass of the exchanged scalar, one looks for solutions of the following generalized eigenvalue problem, obtained by expanding the Nakanishi weights on a suitable basis

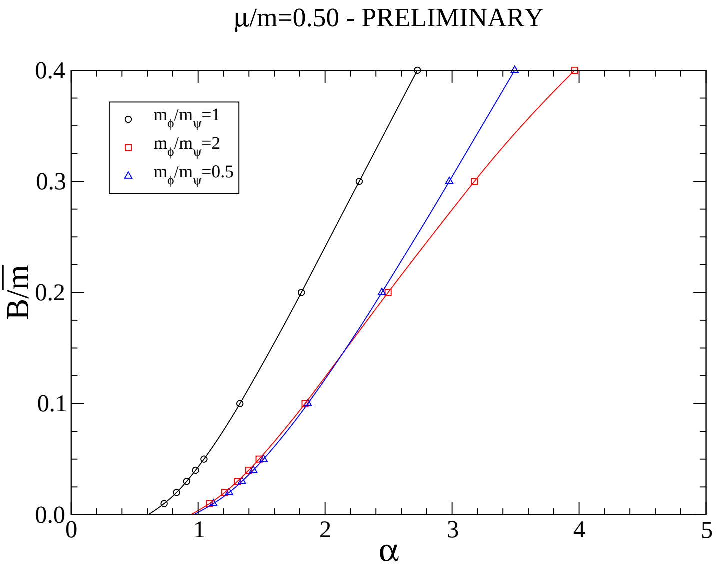
$$B(M)g = \alpha A(M)g$$

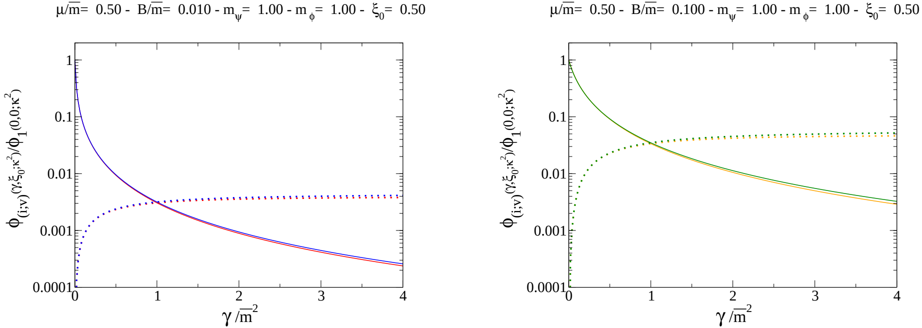
with $\alpha = \lambda_\psi \lambda_\phi / 16\pi m_\phi$

$\mu/\bar{m} = 0.15$

$B/\bar{m}$	$\alpha\,(m_\phi/m_\psi = 1)$	$\alpha\,(m_\phi/m_\psi = 2)$	$\alpha\,(m_\phi/m_\psi = 0.5)$
0.01	0.288	0.385	0.388
0.02	0.361	0.467	0.471
0.03	0.422	0.537	0.540
0.04	0.477	0.600	0.603
0.05	0.528	0.657	0.660
0.10	0.753	0.916	0.914
0.20	1.149	1.386	1.351
0.30	1.524	1.858	1.757
0.40	1.899	2.372	2.152
0.50	2.285	2.974	2.545

We have expanded the Nakanishi weight function in terms of Gegenbauer (z) and Laguerre (γ) polynomials. Good convergence can be achieved with up to 18×18 polynomials and 48 gaussian points.


μ/m=0.50 - PRELIMINARY
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
B/m̄
α
mφ/mψ=1
mφ/mψ=2
mφ/mψ=0.5



Solid lines : $\int \frac{dk^-}{2\pi}\phi_a(k,p)$ Dotted lines: $\gamma^2\left[\int \frac{dk^-}{2\pi}\phi_a(k,p)\right]$

Conclusions & Perspectives

- For the first time a boson-fermion model has been studied within a rigorous field-theoretical framework (the ladder Bethe-Salpeter Equation in Minkowski space).
- The cross-fertilization between the Light-Front framework and the Nakanishi PTIR paves the path toward a new class of non perturbative calculations.
- With more realistic Kernels one can apply the approach to many areas, from investigations of hadron systems (quark-diquark nucleon models) to studies in condensed-matter physics (electron-phonon systems) .