Quantum field theory in two dimensions: light-front versus space-like solutions

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ABSTRACT: A few non-perturbative topics of the quantum field theory in 2D are studied in both the conventional (SL) and light-front (LF) versions. First, we give a concise review of the recently proposed **quantization of the two-dimensional massless LF fields**. The **LF version of bosonization** proves its consistency and efficiency. The bosonized form of the Thirring model follows in a simple and straightforward way. As a further application, we demonstrate the closeness of the 2D massless LF quantum fields to **conformal field theory** (CFT). We calculate several correlation functions including those between the components of the LF energy-momentum tensor, derive the **LF version of the Virasoro algebra** and work out further properties of the massless LF fields. Going over to the Euclidean time, we can immediately transform all calculated quantities to the (anti)holomorphic form. The results found are in agreement with those from CFT. In the final part of our contribution, we show that the proposed framework provides us with the elements needed for an **independent LF study of the exactly solvable models**. We calculate the **non-perturbative**

correlation functions from the exact operator solution of the LF Thirring and Thirring-Wess models and compare it to the analogous results in the SL theory. While the vacuum effects are automatically taken into account in the LF case, the non-trivial vacuum structure has to be incorporated by **an explicit diagonalization of the SL Hamiltonians**, in order to obtain the complete SL solution of the field theories under study.

I. INTRODUCTION

Prevailing opinion: the LF form of QFT has a great potential, but is in some aspects mysterious, maybe incomplete and at least counter-intuitive

Can the LF theory with its drastically simplified vacuum structure generate the same predictions as the SL form?

Important: to clarify areas where the LF theory seemingly fails

Our focus over a few years: understand the massless LF fields in D=1+1, correct quantization

Relevant for: **solvable models** in 2D – independent LF solutions, comparison with the well-established (but not alwas completely correct - true interacting quantum currents, vacuum aspect!) SL solutions (Thirring, Thirring-Wess models)

Here also: relationship to 2D conformal field theory (CFT)

correlation functions of the LF massless fields (elementary or composite - energy-momentum tensor) coincide with the CFT results after going over to the euclidean time $x^0, x^{\pm} \to -i\tau \pm x^1$)

algebra of conformal generators (Virasoro algebra) correctly obtained including its quantum part

The LF operator solution of the Thirring and Thirring-Wess models generates full non-perturbative correlation functions. In the SL solutions, the vacuum state has to constructed (Bogoliubov transformation) in addition to obtain the complete correlation functions

II. MASSLESS LIGHT FRONT FIELDS IN D=1+1

LF notation: $x^\mu = (x^+, x^-) = (x^0 + x^1, x^0 - x^1)$

the momentum k^μ

$$k^\mu = (k^+, k^-), \quad \partial_\pm = \frac{\partial}{\partial x^\pm}, \quad \hat{k} \cdot x = \frac{1}{2}k^+ x^- + \frac{1}{2}\hat{k}^- x^+, \quad k^2 = \mu^2 \Rightarrow \hat{k}^- = \frac{\mu^2}{k^+}. \tag{1}$$

$\hat{k}^-$ is the on-shell LF energy. No sign ambiguity analogous to $E(k^1) = \pm\sqrt{(k^1)^2 + \mu^2}$ of the conventional theory, both k^+, k^- can be taken positive.

A. Scalar field

The LF massless Klein-Gordon equation

$$\partial_+ \partial_- \tilde{\phi}(x) = 0 : \tag{2}$$

has the general solution of the form

$$\tilde{\phi}(x) = \tilde{\phi}(x^+) + \tilde{\phi}(x^-). \qquad (3)$$

The second term obtained from the corresponding massive solution

$$\phi(x) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+) e^{-\frac{i}{2}k^+ x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+} + a^\dagger(k^+) e^{\frac{i}{2}k^+ x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+} \right], \qquad (4)$$

as its massless limit. The first term recovered after the change of variables $k^+ \to \frac{\mu^2}{k^-}$ for $\mu^2 \to 0$. The quantum field expansion and the corresponding algebra then read:

$$\tilde{\phi}(x^-) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+)e^{-\frac{i}{2}k^+x^-} + a^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right], \quad (5)$$

$$\tilde{\pi}(x^-) = -i \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} k^+ \left[a(k^+)e^{-\frac{i}{2}k^+x^-} - a^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right], \quad (6)$$

$$\tilde{\phi}(x^+) = \int_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} \left[\tilde{a}(k^-)e^{-\frac{i}{2}k^-x^+} + \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right], \quad (7)$$

$$\tilde{\theta}(x^+) = -i \int_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} k^- \left[\tilde{a}(k^-)e^{-\frac{i}{2}k^-x^+} - \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right]. \quad (8)$$

$$\lim_{\mu\to 0}\frac{\mu}{k^-}a\Big(\frac{\mu^2}{k^-}\Big)\equiv\tilde{a}(k^-) \tag{9}$$

$$\big[\tilde{a}(k^-),\tilde{a}^\dagger(l^-)\big]=\delta(k^- - l^-),\ \ \big[a(k^-),a^\dagger(l^+)\big]=0. \tag{10}$$

$$\big[\tilde{\phi}(x^-),\tilde{\phi}(y^-)\big]=-\frac{i}{4}\epsilon(x^- - y^-),\ \ \big[\tilde{\phi}(x^+),\tilde{\phi}(y^+)\big]=-\frac{i}{4}\epsilon(x^+ - y^+). \tag{11}$$

Thus, the second half of the solution of the wave equation has been recovered from the massive solution.

Complete consistency: all two-point functions of the massless components coincide with the massless limits of the corresponding two-point functions of the massive fields (NB: they have non-zero limits – the starting observation)

The variables k^+ and k^- coincide - analogous to the SL case $k^0 = |k^1|$. In the LF case $k^- = k^+$ directly (both are positive-definite).

$P^{\pm}$ **must survive the massless limit** - for a consistent quantum theory we have to find the Heisenberg equations

The massless limit of the momentum operator straightforward, the change of variables for the Hamiltonian

Using the Fock commutators (10), the resultant operators

$$P^+ = \int_0^{+\infty} dk^+ k^+ a^\dagger(k^+) a(k^+), \quad P^- = \int_0^{+\infty} dk^+ k^- \tilde{a}^\dagger(k^-) \tilde{a}(k^-) \tag{12}$$

are easily seen to generate the Heisenberg equations

$$2i\partial_+ \phi(x^+) = -[P^-, \phi(x^+)], \quad 2i\partial_- \phi(x^-) = -[P^+, \phi(x^-)], \tag{13}$$

The density $T^{+-}(x)$ vanishes for $\mu = 0$ (conformal symmetry)

B. Fermion field

simpler - no infrared divergencies present. In the massless case, it follows from massive Dirac equation

$$2i\partial_+\psi_2(x) = m\psi_1(x), \quad 2i\partial_-\psi_1(x) = m\psi_2(x) \tag{14}$$

that

$$\psi_2(x) = \psi_2(x^-), \quad \psi_1(x) = \psi_1(x^+). \tag{15}$$

The $\psi_2(x^-)$ component obtained directly as the $m = 0$ limit of the massive $\psi_2(x^+, x^-)$. For $\psi_1(x^+)$, the change of variables done. The field expansion and algebra read

$$\tilde{\psi}_2(x^-) = \frac{1}{\sqrt{4\pi}} \int\limits_0^{+\infty} dp^+ \left[b(p^+) e^{-\frac{i}{2}p^+ x^-} + d^\dagger(p^+) e^{\frac{i}{2}p^+ x^-} \right], \quad (16)$$

$$\tilde{\psi}_1(x^+) = \frac{1}{\sqrt{4\pi}} \int\limits_0^{+\infty} dp^- \left[\tilde{b}(p^-) e^{-\frac{i}{2}p^- x^+} - \tilde{d}^\dagger(p^-) e^{\frac{i}{2}p^- x^+} \right], \quad (17)$$

$$\{b(p^+), b^\dagger(q^+)\} = \{d(p^+), d^\dagger(q^+)\} = \delta(p^+ - q^+), \quad (18)$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-), \quad (19)$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^+)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^+)\} = 0, \quad (20)$$

$$\{\tilde{\psi}_1(x^+), \tilde{\psi}_1{}^\dagger(y^+)\} = \delta(x^+ - y^+), \quad \{\tilde{\psi}_2(x^-), \tilde{\psi}_2{}^\dagger(y^-)\} = \delta(x^- - y^-). \quad (21)$$

The two-point function calculated from the massless $\tilde{\psi}_1(x^+), \tilde{\psi}_2(x^-)$

$$S_{11}^{(+)}(x - y; m = 0) = \frac{1}{2i\pi} \frac{1}{(x^+ - y^+ - i\epsilon^+)}. \quad (22)$$

$$S_{22}^{(+)}(x - y; m = 0) = \frac{1}{2i\pi} \frac{1}{(x^- - y^- - i\epsilon^-)}, \quad (23)$$

coincide with the massless limit of the massive 2-point function.

- A SIMPLE AND CONSISTENT FRAMEWORK ESTABLISHED

No new variables have to be introduced, the massless limit of the massive fields contain the full physical information, quantization consistent

The massless vector current $j^\mu = \overline{\psi}\gamma^\mu\psi$ from the fields (16,17):

$$j^+(x^-) = \lim_{\epsilon^- \to 0} \left[\tilde{\psi}_2^{\dagger}(x^- + \frac{\epsilon^-}{2}) \tilde{\psi}_2(x^- - \frac{\epsilon^-}{2}) + H.c. \right] = 2 : \tilde{\psi}_2^\dagger(x^-)\tilde{\psi}_2(x^-) : \quad (24)$$

$$j^-(x^+) = \lim_{\epsilon^+ \to 0} \left[\tilde{\psi}_1^{\dagger}(x^+ + \frac{\epsilon^+}{2}) \tilde{\psi}_1(x^+ - \frac{\epsilon^+}{2}) + H.c. \right] = 2 : \tilde{\psi}_1^\dagger(x^+)\tilde{\psi}_1(x^+) : \quad (25)$$

Solvable models are based on free Heisenberg fields $\Rightarrow$ the above derivation of the two-dimensional massless LF fermion fields opens the avenue for the genuine light-front solution of this class of models

Before that: bosonization and conformal-symmetry aspects

III. Massless LF bosonization.

In 2D, fermion fields can be represented in terms of boson variables (Coleman, Mandelstam,..).

The massless $\phi(x)$ and $\psi(x)$ fields decompose as

$\phi(x) = \phi(x^+) + \phi(x^-)$, $\psi^T(x) = (\psi_1(x^+), \psi_2(x^-))$

the LF version of bosonization very simple (tilde omitted henceforth)

Start with $\psi_2(x^-)$. Assume that it can be represented as

$$\varphi_2(x^-) = C : e^{i\alpha\phi(x^-)} : \; = Ce^{i\alpha\phi^{(-)}(x^-)}e^{i\alpha\phi^{(+)}(x^-)}. \tag{26}$$

Adjust the constants C and α in such a way, that two φ_2 with different arguments anticommute and $\varphi_2(x^-)$, $\varphi_2^\dagger(y^-)$ satisfy the anticommutation relation (21). The first condition fixes α to the value $\alpha = 2\sqrt{\pi}$. The second condition requires $C = \frac{\lambda e^{\gamma_E}}{4\pi}$. The bosonization formulae read:

$$\hat{\varphi}_2(x^-) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^-)} e^{i2\sqrt{\pi}\phi^{(+)}(x^-)}, \qquad (27)$$

$$\hat{\varphi}_1(x^+) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^+)} e^{i2\sqrt{\pi}\phi^{(+)}(x^+)}, \qquad (28)$$

$$\{\hat{\varphi}_1(x^+), \hat{\varphi}_1(y^+)\} = \delta(x^+ - y^+), \ \{\hat{\varphi}_2(x^-), \hat{\varphi}_2(y^-)\} = \delta(x^- - y^-). \qquad (29)$$

Bosonized vector current: insert the bosonic representation (40) into the hermitean point-split definition(24) to obtain ($1/\epsilon^-$ singularity essential):

$$j^+(x^-) = \frac{2}{\sqrt{\pi}}\partial_-\phi(x^-), \quad j^-(x^+) = \frac{2}{\sqrt{\pi}}\partial_+\phi(x^+). \tag{30}$$

$$\left[j^+(x^-), j^+(y^-)\right] = \frac{i}{\pi}\partial_x\delta(x^- - y^-), \quad \left[j^-(x^+), j^-(y^+)\right] = \frac{i}{\pi}\partial_x\delta(x^+ - y^+). \tag{31}$$

The Schwinger term automatic.

The scalar densities (no singularities - no splitting)

$$\overline{\psi}(x)\psi(x) = \psi_1^\dagger(x^+)\psi_2(x^-) + \psi_2^\dagger(x^-)\psi_1(x^+), \tag{32}$$

$$\overline{\psi}(x)\gamma^5\psi(x) = \psi_1^\dagger(x^+)\psi_2(x^-) - \psi_2^\dagger(x^-)\psi_1(x^+). \tag{33}$$

One obtains ($\phi(x) = \phi(x^+) + \phi(x^-)$)

$$\overline{\psi}(x)\psi(x) = \frac{\lambda e^{\gamma_E}}{4\pi}\cos\left(2\sqrt{\pi}\phi(x)\right), \quad \overline{\psi}(x)\gamma^5\psi(x) = i\frac{\lambda e^{\gamma_E}}{4\pi}\sin\left(2\sqrt{\pi}\phi(x)\right). \tag{34}$$

CONCLUSION: the LF version of bosonization yields the results known from the SL theory, but in a simpler and more transparent form.

Details:

Form the product $\varphi_2(x^-)\varphi_2(y^-)$ and perform the necessary commutations to obtain the opposite order of the operators. This generates the expression

$$\varphi_2(x^-)\varphi_2(y^-) = e^{-\alpha^2\left(D_0^{(+)}(x^- - y^-) - D_0^{(+)}(y^- - x^-)\right)}\varphi_2(y^-)\varphi_2(x^-). \tag{35}$$

The two commutator functions $D_0^{(+)}(\pm(x^- - y^-))$, where

$$D_0^{(+)}(x^- - y^-) = \left[\phi^{(+)}(x^-), \phi^{(-)}(y^-)\right] = \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(x^- - y^- - i\epsilon^-)}, \quad (36)$$

individually diverge, but upon introducing the infrared cutoff λ (cf. Eq.(**??**)) the divergent parts cancel in (35) producing the sign function $\epsilon(x^- - y^-)$. With $\alpha = 2\sqrt{\pi}$, the net result is $e^{i\pi\epsilon(x^- - y^-)} = -1$ for all x^-, y^-. This is the required anticommutativity.

To prove the the second property, form the anticommutator

$$A(x^-, y^-) \equiv \varphi_2(x^-)\varphi_2^\dagger(y^-) + \varphi_2^\dagger(y^-)\varphi_2(x^-) = C^2 \times$$
$$\times \left[e^{4\pi D_0^{(+)}(x^- - y^-)} : \varphi_2(x^-)\varphi_2^\dagger(y^-) : + e^{4\pi D_0^{(+)}(y^- - x^-)} : \varphi_2^\dagger(y^-)\varphi_2(x^-) : \right]. (37)$$

Taking into account the explicit form of the infrared-regularized $D_0^{(+)}$ function

$$D_0^{(+)}(z^-) = -\frac{1}{4\pi} \ln \left[\frac{\lambda}{2} e^{\gamma_E}(iz^- + \epsilon^-)\right], \quad z^- = x^- - y^-, \tag{38}$$

and the fact that two normal-ordered expressions in (37) actually coincide, we find

$$A(x^-, y^-) = \frac{2}{i\lambda e^{\gamma_E}} \left[\frac{1}{z^- - i\epsilon^-} - \frac{1}{z^- + i\epsilon^-}\right] : \varphi_2(x^-)\varphi_2^\dagger(y^-) := \frac{4\pi}{\lambda e^{\gamma_E}} \delta(x^- - y^-). \tag{39}$$

Used: the term in the square bracket is equal to $2i\pi\delta(z^-)$. The operator part on the rhs has reduced to unity due to the presence of this delta-function. It follows that the rescaled operator

$$\hat{\varphi}_2(x^-) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^-)} e^{i2\sqrt{\pi}\phi^{(+)}(x^-)} \tag{40}$$

obeys the correct anticommutation relation (21) and represents the bosonized form of the fermion field $\psi_2(x^-)$. The construction of the second component $\varphi_1(x^+)$ is completely paralel, with $x^- \to x^+$, etc. Thus

$$\hat{\varphi}_1(x^+) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^+)} e^{i2\sqrt{\pi}\phi^{(+)}(x^+)}. \tag{41}$$

The vector current in the bosonic form: Inserting the bosonic form (40) into (24), one gets a product of four exponential operators, which is NOT in the normal order. Commute the two middle terms to normal-order the expression, one generates a term $e^{\alpha^2 D_0^{(+)}(\epsilon^-)}$, which according to (38) behaves as $1/\epsilon^-$. This singularity is canceled by the terms from the exponential, linear in ϵ^-. No vacuum subtractions are needed, since the second (conjugate) term in (24) cancels them automatically. The net result

$$j^+(x^-) = \frac{2}{\sqrt{\pi}}\partial_-\phi(x^-), \quad j^-(x^+) = \frac{2}{\sqrt{\pi}}\partial_+\phi(x^+). \tag{42}$$

The second current component obtained in a completely analogous way.

CONFORMAL SYMMETRY PROPERTIES OF THE MASSLESS LF FIELDS IN D$= 1+1$

Purpose: to elaborate on the properties of the massless LF scalar field for the sake of comparison with the holomorphic conformal formulation

The LF Hamiltonian density $T^{+-}(x)$ of the free massless scalar field $\phi(x) = \phi(x^+) + \phi(x^-)$ vanishes, as required by the conformal symmetry.

The other components of the energy-momentum tensor are nonvanishing:

$$T^{++}(x^-) = 4 : \partial_-\phi(x^-)\partial_-\phi(x^-) :,$$
$$T^{--}(x^+) = 4 : \partial_+\phi(x^+)\partial_+\phi(x^+) : . \tag{43}$$

Here

$$\partial_+\phi(x) = \frac{1}{2i}\int_0^{+\infty} \frac{dk^-\sqrt{k^-}}{\sqrt{4\pi}}\left[a(k^-)e^{-\frac{i}{2}k^-x^+} - a^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right] \tag{44}$$

$$\partial_-\phi(x) = \frac{1}{2i}\int_0^{+\infty} \frac{dk^+\sqrt{k^+}}{\sqrt{4\pi}}\left[a(k^+)e^{-\frac{i}{2}k^+x^-} - a^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right]. \tag{45}$$

The LF Hamiltonian (12) can also be obtained as the x^+-integral of the density $T^{--}(x^+)$, analogously to P^+ which is the x^--integral of $T^{++}(x^-)$.

Compute a few additional correlation functions:

$$\langle 0|\theta(x^+)\theta(y^+)|0\rangle = \frac{1}{\pi}\frac{1}{(x^+ - y^+ - i\delta^+)^2}, \tag{46}$$

$$\langle 0|\pi(x^-)\pi(y^-)|0\rangle = \frac{1}{\pi}\frac{1}{(x^- - y^- - i\delta^-)^2}. \tag{47}$$

Also the correlation functions of the components of the energy-momentum tensor

$$\langle 0|T^{--}(x^+)T^{--}(y^+)|0\rangle = \frac{2}{\pi^2}\frac{1}{(x^+ - y^+ - i\delta^+)^4}, \tag{48}$$

$$\langle 0|T^{++}(x^-)T^{++}(y^-)|0\rangle = \frac{2}{\pi^2}\frac{1}{(x^- - y^- - i\delta^-)^4}. \tag{49}$$

In the holomorphic formulation of the two-dimensional CFT (Belavin, Polyakov, Zamolodchikov 1984, diFrancesco et al. monograph) the Laurent expansion in the variables

$$z = e^{\frac{2\pi}{L}\zeta}, \quad \bar{z} = e^{\frac{2\pi}{L}\bar{\zeta}}, \quad \text{where } \zeta = \tau - ix, \quad \bar{\zeta} = \tau + ix, \tag{50}$$

commonly used. Based on radial quantization that uses the euclidean time τ, with $t \to -i\tau \Rightarrow$ reformulate our results for $\phi(x)$ in a discrete form

First, the massive field in a finite box of lenght $2L$ in x^- or $2T$ in x^+ with periodic boundary conditions

$$\phi(x^+, x^- = -L) = \phi(x^+, x^- = L), \quad \phi(x^+ = -T, x^-) = \phi(x^+ = T, x^-) \tag{51}$$

$\Rightarrow$

$$\phi(x) = \sum_{n=1,2,\ldots} \frac{1}{\sqrt{2Lk_n^+}} \left[a_n e^{-\frac{i}{2}k_n^+ x^- - \frac{i}{2}\frac{\mu^2}{k_n^+}x^+} + a_n^\dagger e^{\frac{i}{2}k_n^+ x^- + \frac{i}{2}\frac{\mu^2}{k_n^+}x^+} \right], \tag{52}$$

$$\phi(x) = \sum_{n=1,2,\ldots} \frac{1}{\sqrt{2Tk_n^-}} \left[a_n e^{-\frac{i}{2}\frac{\mu^2}{k_n^-}x^- - \frac{i}{2}k_n^- x^+} + a_n^\dagger e^{\frac{i}{2}\frac{\mu^2}{k_n^-}x^+ + \frac{i}{2}k_n^- x^+} \right], \tag{53}$$

$$k_n^+ = \frac{2\pi}{L}n, \quad k_n^- = \frac{2\pi}{T}n. \tag{54}$$

The zero mode (the field mode with $n = 0$) vanishes due to the field equation $\mu^2\phi_0(x^+) = 0$ valid in the zero-mode sector. The Fock operators satisfy the algebra

$$\left[a_m, a_n^\dagger\right] = \delta_{m,n}. \tag{55}$$

Performing the massless limit and setting $T = L$:

$$\phi(x^-) = \sum_{n=1,2,\ldots} \frac{1}{\sqrt{2Lk_n^+}} \left[a_n e^{-\frac{i}{2}k_n^+ x^-} + a_n^\dagger e^{\frac{i}{2}k_n^+ x^-}\right], \tag{56}$$

$$\phi(x^+) = \phi_0 + \sum_{n=1,2,\ldots} \frac{1}{\sqrt{2Lk_n^-}} \left[\bar{a}_n e^{-\frac{i}{2}k_n^- x^+} + \bar{a}_n^\dagger e^{\frac{i}{2}k_n^- x^+}\right], \tag{57}$$

$$\left[a_m, a_n^\dagger\right] = \left[\bar{a}_m, \bar{a}_n^\dagger\right] = \delta_{m,n}, \ \left[\bar{a}_m, a_n^\dagger\right] = 0. \tag{58}$$

Alternatively, by a simple discretization of the continuum formulae

Since $\mu = 0$, ϕ_0 can be non-zero. It reduces however to a constant. Its conjugate momentum therefore vanishes.

In a more compact notation:

$$\phi(x^-) = \frac{1}{\sqrt{4\pi}} \sum_{n=\pm1,\pm2,\ldots} \frac{1}{\sqrt{|n|}} a_n e^{-i\frac{\pi}{L}nx^-}, \tag{59}$$

$$\phi(x^+) = \frac{1}{\sqrt{4\pi}} \sum_{n=\pm1,\pm2,\ldots} \frac{1}{\sqrt{|n|}} \bar{a}_n e^{-i\frac{\pi}{L}nx^+}, \tag{60}$$

where

$$a_{-n} \equiv a_n^\dagger, \ \bar{a}_{-n} \equiv \bar{a}_n^\dagger. \tag{61}$$

With the expansions (60,60) one easily calculates the correlation functions of the continuum theory ($z^- = x^- - y^-$):

$$\langle 0|\phi(x^-)\phi(y^-)|0\rangle = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-i\frac{\pi}{L}n(z^- - i\epsilon)} = \quad (62)$$

$$= \frac{1}{4\pi}\ln\left[1 - e^{-i\frac{\pi}{L}(z^- - i\epsilon)}\right] \approx \frac{1}{4\pi}\ln\left[\frac{i\pi}{L}(z^- - i\epsilon)\right], \quad (63)$$

For convergence of the geometrical series, the $i\epsilon$ piece introduced to the exponent. L plays effectively the role of the infared regularization parameter, also the scale to the correlation function (63). L drops out of all the other correlation functions (taking the appropriate derivatives). The results match the continuum results (46) - (49).

The components of the energy-momentum tensor in the discrete representation read, with $K = -\frac{\pi}{L^2}$:

$$T^{++}(x^-) = K \sum_{m,n} \epsilon(m)\epsilon(n)\sqrt{|m||n|} : a_m a_n : e^{-i\frac{\pi}{L}(n+m)x^-}, \tag{64}$$

$$T^{--}(x^+) = K \sum_{m,n} \epsilon(m)\epsilon(n)\sqrt{|m||n|} : \bar{a}_m \bar{a}_n : e^{-i\frac{\pi}{L}(n+m)x^+}. \tag{65}$$

Can be transformed to a ”Virasoro form” by simply taking a Fourier transform. Indeed, assume that $T^{++}(x^-)$ can be represented as

$$T^{++}(x^-) = \frac{1}{4L^2} \sum_{l=0,\pm1,\ldots} L_l e^{-i\frac{\pi}{L}lx^-}. \tag{66}$$

The operator coefficients L_l are then obtained by inverting this relation:

$$L_l = 2L \int_{-L}^{+L} dx^- e^{i\frac{\pi}{L}lx^-} T^{++}(x^-). \tag{67}$$

In particular,

$$L_0 = 4LP^+. \tag{68}$$

Inserting $T^{++}(x^-)$ in the Fock form (64) into (67) gives

$$L_n = -4\pi \sum_{k=\pm 1,\ldots} \epsilon(k)\epsilon(n-k)\sqrt{|k||n-k|}a_k a_{n-k}. \tag{69}$$

A straightforward calculation based on the commutators (58) yields the light front version of the Virasoro algebra, including the c-number term, not present at the classical level (the ”central extension”),

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{70}$$

where c is the ”central charge”. The completely paralel treatment can be given for the component $T^{--}(x^+)$, leading to the algebra (70) with $L_n \to \bar{L}_n$. Obviously, one has $[L_n, \bar{L}_m] = 0$, since $[c_n, \bar{c}_m] = 0$.

Details of the calculation:

Switch back to the ”$a, a^\dagger$” picture ($n > 0$):

$$L_n = -\sum_{k=1}^{n-1}\sqrt{k(n-k)}a_k a_{n-k} + 2\sum_{k=n+1}^{\infty}\sqrt{k(k-n)}a^\dagger_{k-n}a_k, \tag{71}$$

$$L_{-|n|} = -\sum_{k=1}^{n-1} \sqrt{k(n-k)} a_k^\dagger a_{n-k}^\dagger + 2 \sum_{k=n+1}^{\infty} \sqrt{k(k-n)} a_k^\dagger a_{k-n} \quad (72)$$

The "anomaly" comes from the commutator between the first terms:

$$\left[\sum_{l=1}^{m-1} \sqrt{l(m-l)} a_l a_{m-l}, \sum_{k=1}^{n-1} \sqrt{k(n-k)} a_k^\dagger a_{n-k}^\dagger \right] =$$

$$= \sum_{l=1}^{m-1} \sqrt{l(m-l)} \sum_{k=1}^{n-1} \sqrt{k(n-k)} \{ \delta_{m-l,k} \delta_{l,n-k} + \delta_{l,k} \delta_{m-l,n-k} \} =$$

$$= 2\delta_{m,n} \sum_{l=1}^{m-1} l(m-l) = \frac{1}{3} m(m^2-1) \delta_{m,n}. \quad (73)$$

Add factor $1/4$ to correct for different normalization $\Rightarrow$ agreement. All the LF results can be easily transformed to the conformal (holomorphic or antiholomorphic) form by switching to the euclidean time and defining the variables ζ and $\bar{\zeta}$ (50).

Different normalization in the CFT case (factor 2π in the definition of the energy-momentum tensor instead 4 in the LF case). We get (cf. Eq.(47)):

$$\langle 0|\pi(\zeta)\pi(\zeta')|0\rangle = -\frac{1}{(\zeta-\zeta')^2}, \tag{74}$$

$$\langle 0|T(\zeta)T(\zeta')|0\rangle = \frac{c}{2}\frac{1}{(\zeta-\zeta')^4},\ \ c=1. \tag{75}$$

Our field expansions (60) now read

$$\phi(\zeta) = \frac{1}{\sqrt{4\pi}}\sum_{n=\pm1,\pm2,\ldots}\frac{1}{\sqrt{|n|}}a_n z^n,\ \ [a_m,a_n]=\delta_{m+n,0}, \tag{76}$$

$$\phi(\bar{\zeta}) = \frac{1}{\sqrt{4\pi}} \sum_{n=\pm1,\pm2,\ldots} \frac{1}{\sqrt{|n|}} \bar{a}_n \bar{z}^n, \quad [\bar{a}_m, \bar{a}_n] = \delta_{m+n,0}. \tag{77}$$

Analogous to the transition (see diFrancesco, e.g.) to the conformal field in the conventional treatment. The creation and annihilation operators are left unchanged in the transition. This implies in particular that the vacua $|0\rangle$, defined by $a_n|0\rangle = 0$, of the free LF and conformal field theories coincide.

A completely paralel treatment for the fermion field

IV. THE THIRRING MODEL

genuine LF analysis: explicit solution of the field equation

simplifications w.r. to the SL case (no Bogoliubov transformation, physical vacuum = Fock vacuum)

Classical Lagrangian density and the Euler-Lagrange eqs. are

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi + \frac{1}{2}gJ_\mu J^\mu, \quad i\gamma^\mu\partial_\mu\Psi = -g\gamma_\mu J^\mu, \quad J^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x). \tag{78}$$

The LF field equation decomposes into two equations:

$$2i\partial_+\Psi_2 = -gJ^-\Psi_2, \;\; 2i\partial_-\Psi_1 = -gJ^+\Psi_1, \quad J^+ = 2\Psi_2^\dagger\Psi_2, \;\; J^- = 2\Psi_1^\dagger\Psi_1. \tag{79}$$

Introduce the potentials $J(x)$ and $J(x^{\pm})$ (details below):

$$J(x) = -(J(x^{+}) + J(x^{-})),$$
$$J^{+}(x^{-}) = -\frac{2}{\sqrt{\pi}}\partial_{-}J(x), \quad J^{-}(x^{+}) = -\frac{2}{\sqrt{\pi}}\partial_{+}J(x), \tag{80}$$

The Eqs.(79) are solved by

$$\Psi_1(x) = e^{i\frac{g}{\sqrt{\pi}}J(x)}\psi_1(x^{+}), \quad \Psi_2(x) = e^{i\frac{g}{\sqrt{\pi}}J(x)}\psi_2(x^{-}). \tag{81}$$

Quantum solution (normal-ordering):

$$\Psi_1(x) = Z^{-1/2}(\epsilon)e^{i\frac{g}{\sqrt{\pi}}J^{(-)}(x+\frac{\epsilon}{2})+i\frac{g}{\sqrt{\pi}}J^{(+)}(x-\frac{\epsilon}{2})}\psi_1(x^{+}) = \tag{82}$$
$$= e^{i\frac{g}{\sqrt{\pi}}J^{(-)}(x+\frac{\epsilon}{2})}e^{i\frac{g}{\sqrt{\pi}}J^{(+)}(x-\frac{\epsilon}{2})}\psi_1(x^{+}), \tag{83}$$

$$\Psi_2(x) = Z^{-1/2}(\epsilon) e^{i\frac{g}{\sqrt{\pi}} J^{(-)}(x+\frac{\epsilon}{2}) + i\frac{g}{\sqrt{\pi}} J^{(+)}(x-\frac{\epsilon}{2})} \psi_2(x^+) = \tag{84}$$

$$= e^{i\frac{g}{\sqrt{\pi}} J^{(-)}(x+\frac{\epsilon}{2})} e^{i\frac{g}{\sqrt{\pi}} J^{(+)}(x-\frac{\epsilon}{2})} \psi_2(x^+), \tag{85}$$

The interacting currents calculated from the solutions (81) by means of the (hermitean) point-splitting. Using

$$\psi_2^\dagger(x^- + \epsilon^-/2)\psi_2(x^- - \epsilon^-/2) =: \psi_2^\dagger(x^-)\psi_2(x^-) : + V(\epsilon^-),$$

$$\psi_1^\dagger(x^+ + \epsilon^+/2)\psi_1(x^+ - \epsilon^+/2) =: \psi_1^\dagger(x^+)\psi_1(x^+) : + V(\epsilon^+),$$

$$V(\epsilon^\pm) = \frac{1}{4\pi} \int_0^\infty \mathrm{d}p^- e^{-\frac{i}{2}p^\mp(\epsilon^\pm - i\eta)} = -\frac{i}{2\pi} \frac{1}{\epsilon^\pm - i\eta}, \tag{86}$$

one obtains

$$J^+(x) = G(g) j^+(x^-), \;\; J^-(x) = G(g) j^-(x^+), \;\; G(g) = \Big(1 - \frac{g}{2\pi}\Big)^{-1}, \quad (87)$$

i.e. the interacting vector current is a ”renormalized” free current.

Convenient to **bosonize the currents** by a Fourier transformation:

$$j^+(x^-) = -\frac{i}{\sqrt{\pi}} \int\limits_0^\infty \frac{dk^+ \, k^+}{\sqrt{4\pi k^+}} \big[c(k^+) e^{-\frac{i}{2} k^+ x^-} - H.c. \big], \; c(k^+) = \frac{i \hat{c}(k^+)}{2\sqrt{k^+}},$$

$$\big[c(k^+), c^\dagger(l^+) \big] = \delta(k^+ - l^+), \quad (88)$$

$$\hat{c}(k^+) = \int\limits_0^\infty ds^+ \big[b^\dagger(s^+) b(s^+ + k^+) - (b \to d) + d(s^+) b(k^+ - s^+) \theta(k^+ - s^+) \big].$$

Similarly

$$j^-(x^+) = -\frac{i}{\sqrt{\pi}} \int_0^\infty \frac{dk^-\ k^-}{\sqrt{4\pi k^-}} \left[a(k^-)e^{-\frac{i}{2}k^- x^+} - H.c.\right]. \tag{89}$$

With an implicit infrared regularization, the **potentials ("integrated currents")** are expressed as

$$j(x^-) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \left[c(k^+)e^{-\frac{i}{2}k^+ x^-} + c^\dagger(k^+)e^{\frac{i}{2}k^+ x^-}\right],$$

$$j(x^+) = \int_0^\infty \frac{dk^-}{\sqrt{4\pi k^-}} \left[a(k^-)e^{-\frac{i}{2}k^- x^+} + a^\dagger(k^-)e^{\frac{i}{2}k^- x^+}\right], \tag{90}$$

i.e. they are simply two components of a massless LF scalar field $j(x)$, which is related to the interacting potential by $J(x) = G(g)j(x)$.

NEXT: calculate n-point functions, their Fourier transform will give information about particle spectrum

Two-point function $\langle vac|\Psi(x)\overline{\Psi}(y)|vac\rangle$: Fourier transform: $\frac{p^+}{\lambda}$ terms appear. No poles $\Rightarrow$ no asymptotic states.

Preliminary comparison: **LF results more complex than the SL ones!** Indication of the missing element (non-trivial vacuum) in the SL solution:

the unitary operator that diagonalizes the SL Hamiltonian can be used to transform all operators to a new representation while keeping the Fock vacuum as the physical vacuum (two alternative views) - the probable source of remedy (and agreement with the LF results)

Summary and conclusions

- A consistent light-front quantization of the 2-D massless scalar and fermion fields
- Bosonization simple
- Conformal-symmetry aspects built-in, agreement with CFT
- Independent study of the solvable models within the LF field theory
- Non-perturbative correlation functions, preliminary LF results richer, vacuum in the SL treatment plays an important role
- Confirms simplicity and power of the LF QFT (Dirac)