Euclidean relativistic quantum mechanics - scattering asymptotic conditions

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Outline

- Motivation for the Euclidean approach
- Euclidean formulation of relativistic quantum theory
- Scattering - general considerations
- The one-body problem - strong asymptotic condition
- Existence of S-matrix, open problems.

Why use a Euclidean Approach?

- **Can do Minkowski quantum mechanics directly in the Euclidean representation without analytic continuation.**
- **Cluster are properties easy to satisfy compared with direct-interaction models.**
- **Direct connection: Euclidean $\to$ light-front .**
- **Direct connection to Lagrangian phenomenology.**
- **Locality can be relaxed without destroying spectral condition, cluster properties or relativistic invariance.**
- **Scattering without analytic continuation possible.**

What is the Euclidean approach?

There are two different analytic continuations of Euclidean Green functions

- **Euclidean Green functions $\rightarrow$ time ordered Green functions (Schwinger)**

$$G_n^e(x_{1e} \cdots x_{ne}) \rightarrow \langle 0| T(\phi_1(x_1) \cdots \phi_n(x_n))|0\rangle$$

- **Euclidean Green functions $\rightarrow$ vacuum expectation values of products of fields = quantum mechanics (Osterwalder and Schrader)**

$$G_n^e(x_{1e} \cdots x_{ne}) \rightarrow \langle 0|\phi_1(x_1) \cdots \phi_n(x_n)|0\rangle$$

Schwinger continuation

$$G_{nm}(x_1, \cdots, x_n) = \lim_{\theta \to \pi/2} G_{ne}(x_1(\phi), \cdots, x_n(\phi))$$

$$x_k(\phi) = (x_{ke}^0 e^{i\phi}, \mathbf{x}_k) \qquad 0 \leq \phi < \pi/2$$

Osterwalder - Schrader continuation

$$\langle 0|\phi_1(x_1) \cdots \phi_n(x_n)|0\rangle = \lim_{0 < x_{1e}^0 < \cdots < x_{ne}^0 \to 0} G_{ne}(x_{1e}^0 + it_1, \mathbf{x}_1, \cdots x_{ne}^0 + it_n, \mathbf{x}_n)$$

Euclidean time ordering = field ordering

Osterwalder & Schrader: analytic continuation not necessary - relativistic quantum mechanics can be formulated directly in the Euclidean representation

Form of the Minkowski Hilbert space $\mathcal{H}$ inner product in the Euclidean representation

$$\langle \psi | \phi \rangle =$$

$$\sum_{mn} \int \psi_m^*(T_e x_{1e}, \cdots, T_e x_{me}) G^e_{m+n}(x_{me}, \cdots, x_{1e}, y_{1e}, \cdots, y_{ne}) \times$$

$$\phi_n(y_{1e}, \cdots, y_{ne}) d^{4m}x d^{4n}y$$

- $T_e =$ **Euclidean time reflection**
- $\psi_m, \phi_n = 0$ **unless** $0 < x^0_{e1} < x^0_{e2} < x^0_{e3} \cdots$
- **Reflection positivity (essential property of G_n)** $\langle \psi | \psi \rangle \geq 0$

Relativistic Invariance

Poincaré Lie Algebra on $\mathcal{H}$

$$H\psi_n(x_{1e}, x_{2e}, \cdots, x_{ne}) = \sum_{k=1}^{n} \frac{\partial}{\partial x_{ke}^0} \psi_n(x_{1e}, x_{2e}, \cdots, x_{ne})$$

$$\mathbf{P}\psi_n(x_{1e}, x_{2e}, \cdots, x_{ne}) = -i \sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_{ke}} \psi_n(x_{1e}, x_{2e}, \cdots, x_{ne})$$

$$\mathbf{J}\psi_n(x_{1e}, x_{2e}, \cdots, x_{ne}) = -i \sum_{k=1}^{n} \mathbf{x}_{ke} \times \frac{\partial}{\partial \mathbf{x}_{ke}} \psi_n(x_{1e}, x_{2e}, \cdots, x_{ne})$$

$$\mathbf{K}\psi_n(x_{1e}, x_{2e}, \cdots, x_{ne}) = \sum_{k=1}^{n} (\mathbf{x}_{ke} \times \frac{\partial}{\partial x_{ke}^0} - x_{ke}^0 \frac{\partial}{\partial \mathbf{x}_{ke}}) \psi_n(x_{1e}, x_{2e}, \cdots, x_{ne}).$$

$$\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$$

Self-adjoint in the Euclidean representation of the Hilbert space.

- **Satisfy Poincaré commutation relations.**

- **Satisfy cluster properties provided G_e has a cluster expansion.**

Comments

$$M^2 = H^2 - \mathbf{P}^2 \qquad P^{\pm} = H \pm P^3$$

- **are simple differential operators on** $\mathcal{H}$
- **Dynamics is in the kernel of the scalar product**
- **Hilbert space vectors are equivalence classes**
  $f \equiv g \qquad \Longleftrightarrow \qquad < f - g | f - g >= 0$
- **Reflection positivity is not automatic - a general structure theorem is not known**
- **Analytic continuation not used - Poincaré generators and wave functions use Euclidean variables**
- **Reflection positivity** $\Rightarrow \quad H \geq 0$

Scattering in the Euclidean representation

Cluster Properties

$$\lim_{\mathbf{a}\to\infty}(G^e_{m+n}(X_{me}+\mathbf{a}, Y_{ne}, \cdots) - G^e_m(X_{me})G^e_n(Y_{ne})) \to 0$$

$$X_{me} = (x_{1e}, \cdots, x_{me})$$

- **Hilbert space + Poincaré generators satisfying cluster properties $\Rightarrow$ scattering theory should be possible.**
- **Apply methods used in non-relativistic mutlichannel scattering**
  - **Strong asymptotic condition**
  - **Cook's condition**
- **How to proceed in this abstract setting?**

Scattering asymptotic condition

$$\lim_{t\to\pm\infty} \| e^{-iHt}|\psi_\pm\rangle - \Phi e^{-iH_{12}t}|\chi_{12}\rangle \| = 0$$

where

$$\Phi e^{-iH_{12}t}|\chi_{12}\rangle := \sum\!\!\!\!\!\!\int \underbrace{|\phi_{b1},\mathbf{p}_1,\mu_1,\phi_{b2},\mathbf{p}_2,\mu_2\rangle}_{\Phi} \times$$

$$\underbrace{e^{-i(E_1(\mathbf{p}_1)+E_2(\mathbf{p}_2))t}}_{e^{-iH_{12}t}} d\mathbf{p}_1 d\mathbf{p}_2 \underbrace{\chi_1(\mathbf{p}_1,\mu_1)\chi_2(\mathbf{p}_1,\mu_1)}_{\textit{wave packets}}$$

and

$$M_{bi}|\phi_{bi}\rangle = m_{b_i}|\phi_{bi}\rangle \qquad E_i(\mathbf{p}_i) = \sqrt{m_{b_i}^2 + \mathbf{p}_i^2}$$

$|\phi_{bi}\rangle$ **are one-body (bound-state) solutions - associated with subsystem G_{ne} in cluster expansion**

Quasilocal fields (Minkowski field theory)

$$\Phi_I(x) \overleftrightarrow{\partial} \xi_m(x) \to \Phi_Q(x) \overleftrightarrow{\partial} \xi_m(x)$$

- $\Phi_I(x) =$ **interpolating field**
- $\xi_m(x) =$ **positive energy solution of Klein-Gordon equation**

$$\Phi_Q(x) := \int e^{-i(x-y)\cdot p} f_m(p^2) \Phi_I(y) \frac{d^4 p d^4 y}{(2\pi)^4}$$

$$f_m(p^2) = \begin{cases} 1 & -p^2 = m^2 \\ 0 & -p^2 \in \sigma(\text{other intermediate states}) \end{cases}$$

$$\Phi_Q(x)|0\rangle = \text{one-particle mass eigenstate}$$

Replacing $\Phi_I(x)$ by $\Phi_Q(x)$ in scattering asymptotic condidtion gives strong limits (Haag Ruelle scattering)

Scattering probabilities

$$P_{fi} = |\langle\psi_+|\psi_-\rangle|^2 = |\langle\chi_{12f}|S|\chi_{12i}\rangle|^2$$

Sufficient condition for existence of $|\psi_\pm\rangle$
(convergence of limit on last slide)

Cook's condition

$$\int_a^\infty \|(H\Phi - \Phi H_{12})e^{\mp iH_{12}t}|\chi_{12}\rangle\| dt < \infty$$

Proof requires one-body solutions

Tasks

- **How do we find one-body solutions in the Euclidean representation ?**
- **How do we verify Cook's condition in the Euclidean representation ?**
- **How do we calculate cross sections?**

What is a one-body solution?

Find normalizable $\psi \in \mathcal{H}$ such that

$$0 = \langle \xi | (M^2 - m^2) | \psi \rangle =$$

$$\sum \!\!\!\!\!\!\! \int \xi^*_m(x_{me}, \cdots, x_{1e}) G^e_{m+n}(T_e x_{me}, \cdots, T_e x_{1e}, y_{1e}, \cdots, y_{ne}) \times$$

$$(M^2 - m_b^2)\psi_n(y_{1e}, \cdots, y_{ne}) d^{4m}x d^{4n}y = 0$$

for all $\xi \in \mathcal{H}$.

- **ψ_n and ξ_m satisfy the relative time support condition**
- **M^2 is a differential operator**
- **Different ψs can represent the same one-body state**

Simplest case $(2 \to 2)$

Cluster properties

$$G_{4e}(x_{1e}, x_{2e}, y_{2e}, y_{1e}) = G_{2e}(x_{1e}, y_{1e})G_{2e}(x_{2e}, y_{2e}) + \underbrace{G^c_{4e}(x_{1e}, x_{2e}, y_{2e}, y_{1e})}_{\textit{connected part}}$$

General form of G_{2e} given by the Lehmann representation

$$G_{2e}(\mathrm{x}_e, \mathrm{y}_e) = \frac{1}{(2\pi)^4} \int \frac{e^{ip\cdot(x_e - y_e)}\rho(m)}{m^2 + p_e^2} d^4p_e dm$$

Lehmann weight

$$\rho(m) = \sum_k \underbrace{z_k \delta(m - m_k)}_{\text{one-body parts}} + \rho_c(m)$$

$$M^2 = \nabla^2_{xe} = \frac{\partial^2}{\partial x_e^{02}} + \nabla^2_{\mathbf{x}}$$

One-body problem:

For fixed m_k find $\psi_{1b}(y_e)$ so

$$\langle \xi | M^2 | \psi \rangle = (\xi, T_e G_{2e} M^2 \psi)_e =$$

$$\int \xi^*(x_e) \frac{d^4x_e d^4y_e}{(2\pi)^4} \int \frac{e^{ip\cdot(T_e x_e - y_e)} \rho(m)}{m^2 + p_e^2} d^4p_e dm \nabla^2_{ye} \psi_{1b}(y_e) =$$

$$\int \xi^*(x_e) \frac{d^4x_e d^4y_e}{(2\pi)^4} \int \frac{e^{ip\cdot(T_e x_e - y_e)} z_k \delta(m - m_k)}{m^2 + p_e^2} d^4p_e dm m_k^2 \psi_{1b}(y_e)$$

holds for all $\xi(x_e)$

where $\xi(x_e), \psi_{1b}(x_e)$ vanish for $x_e^0 \leq 0$

Structure of the Hilbert space inner product

$$\langle \xi | \psi \rangle = \int \xi^*(x_e) G_{2e}(T_e x_e - y_e) \psi(y_e) d^4 x_e d^4 y_e =$$

$$\int \frac{d^4 p_e d^4 x_e d^4 y_e}{(2\pi)^4} \xi^*(x_e) \frac{e^{i p_e^0 (-x_e^0 - y_e^0) + i \mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \rho(m)}{(p_e^0)^2 + \mathbf{p}^2 + m^2} \psi(y_e) dm =$$

Use support condition to close p_e^0 contour in LHP

$$\int \underbrace{\xi^*(x_e) \frac{d^4 x_e}{(2\pi)^{3/2}} e^{-\omega_m(\mathbf{p}) x_e^0 + i \mathbf{p} \cdot \mathbf{x}}}_{\tilde{\xi}^*(\mathbf{p}, m)} \frac{\rho(m) dm d\mathbf{p}}{2\omega_m(\mathbf{p})} \underbrace{e^{-\omega_m(\mathbf{p}) y_e^0 - i \mathbf{p} \cdot \mathbf{y}} \frac{d^4 y_e}{(2\pi)^{3/2}} \psi(y_e)}_{\tilde{\psi}(\mathbf{p}, m)}$$

Lorentz invariant measure - no analytic continuation!

Structure of M^2 - integrate by parts

$$\int \frac{\rho(m)dm}{2\omega_m(\mathbf{p})} e^{-\omega_m(\mathbf{p})y_e^0 - i\mathbf{p}\cdot\mathbf{y}} \frac{d^4 y_e}{(2\pi)^{3/2}} \nabla_{ye}^2 \psi(y_e) =$$

$$\int \frac{\rho(m)dm}{2\omega_m(\mathbf{p})} e^{-\omega_m(\mathbf{p})y_e^0 - i\mathbf{p}\cdot\mathbf{y}} \frac{d^4 y_e}{(2\pi)^{3/2}} m^2 \psi(y_e)$$

Choose $f(m^2)$ **so** $f(m^2)\rho(m) = z_k \delta(m - m_k)$

$f(\nabla_{ye}^2) \to f(m^2)$ **but** $f(\nabla_{ye}^2)\psi(y_e) = 0$ **for** $y_e^0 < 0$**?**

$f(z)$ **cannot be analytic - must vanish on the support of** $\rho_c(m)$**. If satisfied the one-body solution is**

$$\psi_{1b}(x_e) = f(\nabla_{xe}^2)\psi(x_e)$$

$f(\nabla^2_{ye})$ with infinite numbers of derivatives can change support conditions

Can map ψ out of physical Hilbert space !

for example

$$e^{a\frac{d}{dx}} g(x) = g(x+a)$$

Polynomial $f(\nabla^2_{ye})$ preserves the support condition; but $f(z)$ does not vanish on any finite interval.

Why is the Euclidean relative time support constraint needed?

Negative norm states possible in the absence of constraints; eg.

$$T_e\psi = -\psi \quad \to \quad \langle \psi | \psi \rangle < 0$$

In addition for several variables the Euclidean time supports must be disjoint

- **Needed for support condition to be preserved under cluster limits**
- **Provides exponential fall off for t_e**
- **The order is not important (locality = symmetry)**

$$G_{ne}(x_1, \cdots, x_n) = G_{ne}(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

Polynomial approximations to $f_k(m^2)$ possible for unphysical cases

Discrete Lehmann weight (not physical)

$$f_k(m^2) = \prod_{i \neq k}^{N} \frac{m^2 - m_i^2}{m_k^2 - m_i^2}$$

Continuous part of Lehmann weight has compact support (not physical)

$$f_k(m^2) \approx p_k(m^2)$$

Polynomial approximations converge uniformly to any continuous function with compact support (Weierstrass).

What do we know about $\rho(m)$?

- **The physical Lehmann weight is expected to have support on the half line, with the continuous spectrum extending out to $+\infty$.**

- **Axioms $\Rightarrow$ $\rho_c(m)$ is polynomially bounded.**

- **After closing the p_e^0 contour in G_{2e}, the residue has the factor $e^{-\omega_m(\mathbf{p})y_e^0}$. For $\rho(m)$ with a mass gap this provides an exponential suppression with respect to mass.**

- **If polynomials in m^2 are complete on the half line with respect to this weight, $f(m^2)\psi = f(\nabla_e^2)\psi$ is a one-body eigenstate satisfying the support condition.**

How do we establish completeness?

Standard proofs for Laguerre or Hermite polynomials specialized - not easy to generalize.

Stiltjes moment problem - when do a set of moments on the half line determine a unique measure, $w(x)dx$?

Sufficient condition - Carleman condition

$$\gamma_n = \int_0^\infty w(x)dxx^n \qquad \sum_{n=1}^{\infty} \gamma_n^{-\frac{1}{2n}} > \infty$$

$$\Downarrow$$

Orthogonal polynomials in x with weight $w(x)$ are complete on $[0, \infty]$.

Comments on Carleman condition

- If the Carleman condition is satisfied, it is still satisfied by removing a finite number of monomials. The remaining monomials are still complete!

- For one-body solutions we need completeness of polynomials in m^2 rather than m

- The Stiltjes moment problem - multiplication by x is an operator; the uniqueness of the measure is related to whether x has a unique self adjoint extension on the Hilbert space of polynomials.

We have a solution, $f_k(\nabla^2_{xe})\psi(x_e) = \lim p_n(\nabla^2_{xe})\psi(x_e)$, to the one-body problem that preserves the Euclidean time support condition if polynomials in m^2 are complete with respect to weight:

$$w(m) := \int \psi(\mathbf{p}, x_0) \frac{e^{-\omega_m(\mathbf{p})(x_{0e}+y_{0e})}}{2\omega_m(\mathbf{p}^2)} \rho(m^2)\psi(\mathbf{p}, y_{0e}) d\mathbf{p} dx_0 dy_0$$

$$\gamma_n := (\psi, T_e G_{2e}(\nabla^2)^n \psi) = \int_0^\infty w(m) m^{2n}$$

Completeness follows if

$$\sum_{n=1}^{\infty} \gamma_n^{-\frac{1}{2n}} > \infty$$

Check of Carleman condition
(physical case support of $\rho(m) \subset [0,\infty]$)

Axioms $\rightarrow \rho(m) < Cm^k$ **(polynomially bounded)**

$$\gamma_n := \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}} \rho(m) = m^{2n} dm$$

$$\gamma_n \rightarrow \gamma'_n = \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}} m^{2n+k} dm =$$

$$\frac{1}{2}\int_0^\infty \frac{e^{-p\tau\cosh(\eta)}}{\cosh(\eta)} (p\sinh(\eta))^{2n+k} \cosh(\eta) d\eta \leq \frac{1}{2}\tau^{-2n-k}\Gamma(2n+k-2)$$

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x+\theta/12x} \qquad \theta \in [0,1]$$

$$(\frac{1}{\gamma_n})^{\frac{1}{2n}} \geq \sqrt{\frac{2}{\pi}}^{\frac{1}{2n}} \tau^{1+\frac{k}{2n}} (2n+k-2)^{-1-(k-3/2)/2n} e^{(1+(k-2)/2n)-\frac{\theta}{1+(k-2)/2n}}$$

$$\sum_{n=1}^{\infty} \gamma_n^{-\frac{1}{2n}} > \sum_{n=0}^{\infty} \frac{c}{2n+k-2} > \infty$$

The inequality shows that $f_k(M^2) = f_k(\nabla_{xe}^2) = \lim p_n(\nabla_{xe}^2)$ with desired support properties can be approximated by a polynomial with controlled errors.

It ensures the existence of projections onto one-body solutions in the Euclidean representation of Hilbert space.

Scattering - Euclidean case $(2 \to 2)$

Role of one-body solutions in the Cook condition

Cluster properties leads to the decomposition

$$G_{4e}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_2, \mathrm{y}_1) = G_{2e}(\mathrm{x}_1, \mathrm{y}_1) G_{2e}(\mathrm{x}_2, \mathrm{y}_2) + G_{4ec}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_2, \mathrm{y}_1)$$

$$G_{2e}(\mathrm{x}_1, \mathrm{y}_1) = \frac{1}{(2\pi)^4} \int \frac{e^{ip\cdot(x-y)}\rho(m)}{m^2 + p^2} d^4 p dm$$

$$\rho(m) = \sum_i z_i \delta(m - m_i) + \rho_c(m)$$

One-body solutions

$$\langle x_1, x_2 | \Phi | \mathbf{p}_1, \mathbf{p}_2 \rangle = f_1(\nabla_1^2) f_2(\nabla_2^2) \delta(\mathrm{x}_1^0 - \tau_1) \delta(\mathrm{x}_2^0 - \tau_2) \frac{1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2}$$

Cluster properties of Euclidean Green functions essential for scattering.

$\psi(x) = \xi(\mathbf{x})\delta(x^0 - \tau)$ **is square integrable on** $\mathcal{H}$ **in spite of the delta function!**

Provides a simple way to satisfy the support condition.

We use the fact that m^2 **on the two-point function is represented by the Euclidean 4-Laplacian.**

$f_i(\nabla_e^2)$ **selects mass of scattering asymptotes (one-body solution).**

$f(\nabla^2) = \lim p_n(\nabla^2)$ **preserves the support condition.**

Large equivalences class representing one-body solutions means that solutions be extracted from few-point functions.

Existence (Role of one-body solutions in establishing Cook's condition - Euclidean representation)

$$\int_0^\infty \|(H\Phi - \Phi H_f)e^{-iH_f t}|\chi_{0\pm}(0)\rangle\| < \infty$$

$$\|(H\Phi - \Phi H_f)e^{-iH_f t}|\chi_{0\pm}(0)\rangle\|^2 =$$

$$(\chi_f, e^{iH_f t}(\Phi^\dagger H - H_f \Phi^\dagger), T_e(G_{2e}G_{2e} + G_{4ec})(H\Phi - \Phi H_f)e^{-iH_i t}\chi_i)$$

$$\|(H\Phi - \Phi H_f)e^{iH_f t}|\chi_0\rangle\|^2 =$$

$$\int \chi_1^*(\mathbf{p}_1)\chi_2^*(\mathbf{p}_2)e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t}d\mathbf{p}_1 d\mathbf{p}_2$$

$$(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2))\langle \mathbf{p}_1, \mathbf{p}_2|\Phi^\dagger|x_1, x_2\rangle \times$$

$$d^4x_1 d^4x_2 (G_{2e}(T_e x_1, y_1)G_{2e}(T_e x_2, y_2) + G_{4ec}(T_e x_1, T_e x_2, y_2, y_1))d^4y_1 d^4y_2 \times$$

$$(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2'))\langle y_1, y_2|\Phi|\mathbf{p}_1', \mathbf{p}_2'\rangle$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t}\chi_1(\mathbf{p}_1')\chi_2(\mathbf{p}_2')d\mathbf{p}_1' d\mathbf{p}_2'$$

- **Integrate by parts**

$$\frac{\partial}{\partial x^0} \to \omega_m(\mathbf{p})$$

- $f_i(m^2)$ **in** Φ **picks out single** m_i **so the** G_{2e} **terms do not contribute**

- **The strong limit would not exist without the** $f_i(\nabla^2)$ **factors.**

- **This eliminates the Maiani Testa problem!**

What remains is

$$\|(H\Phi - \Phi H_f)e^{iH_f t}|\chi_0\rangle\|^2 =$$

$$\int \chi_1^*(\mathbf{p}_1)\chi_2^*(\mathbf{p}_2)e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t}d\mathbf{p}_1 d\mathbf{p}_2$$

$$(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2))\langle \mathbf{p}_1, \mathbf{p}_2|\Phi^\dagger|x_1, x_2\rangle \times$$

$$d^4x_1 d^4x_2 G_{4ec}(T_e x_1, T_e x_2, y_2, y_1) d^4y_1 d^4y_2 \times$$

$$(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2'))\langle y_1, y_2|\Phi|\mathbf{p}_1', \mathbf{p}_2'\rangle$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t}\chi_1(\mathbf{p}_1')\chi_2(\mathbf{p}_2')d\mathbf{p}_1' d\mathbf{p}_2'$$

Integrand $\sim t^{-3}$ for good S_{4c}

Conclusions -outlook

- Possible to do ordinary quantum mechanics using Euclidean Green functions without explicit analytic continuation
- A Euclidean generalization of Haag Ruelle scattering is possible. (interpolating fields replaced by exact asymptotic one-body solutions)
- In previous work we have demonstrated the convergence of some computational methods for scattering. Other methods are possible.
- Euclidean $\rightarrow$ light front?
- Reflection positivity?
- QCD?

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