Two-fermion Bethe-Salpeter Equation in Minkowski Space

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de Paula & FSV arXiv: 1609.00868: two-fermion bound system

FSV PRD 85 (2012) 036009: general formalism for bound and scattering states

FSV PRD 89 (2014) 016010: bound states and LF momentum distributions for two scalars

FSV EPJC 75 (2015) 398: scattering lengths for two scalars

Gutierrez et al PLB 759 (2016) 131: spectra of excited states and LF momentum distributions

Outline

1. Motivations and generalities: BS Amplitude and BS Equation for a two-fermion bound system $\to \mathcal{L} = g\bar{\psi}\Gamma\psi\ \chi$
2. Nakanishi integral representation (NIR) and the BS Amplitude
3. The exact projection of the BSE onto the null plane and the NIR of BSA
4. Spin dof and BSE
5. Conclusions & Perspectives

Motivations and tools

- To achieve a fully covariant description for a two-fermion system, in Minkowski space, through the non perturbative framework yielded by the Bethe-Salpeter equation (BSE)
- To determine from the BS amplitude, directly in Minkowski space, the relevant momentum distributions
- The fermionic nature of the constituents is suitably managed within the Light-front (LF) framework, making more simple the numerical calculations
- Pivotal role of the Nakanishi Integral Representation (NIR) of the BS amplitude

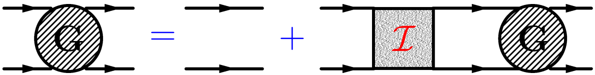
Recall: the recognized non perturbative approach in field theory is the lattice, but it is played in Euclidean space

The BSE in a nutshell

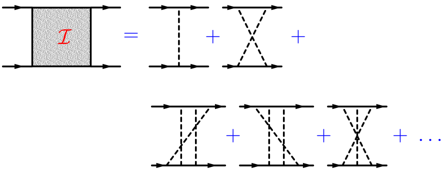
The 4-point Green's Function,

$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 > \quad ,$$

fulfills an integral equation $G = G_0 \; + \; G_0 \, \mathcal{I} \; G$



$\mathcal{I}$ $\equiv$ kernel given by the infinite sum of irreducible Feynman graphs



All the expected contributions from iterations

Insert a complete Fock basis in

$$G(x_1, x_2; y_1, y_2) =< 0 \,|\, T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} \,|\, 0 >$$

then in the Fourier space, the bound state contribution (assuming only one non degenerate bound state for the sake of simplicity) appears as a pole, i.e.

$$G_B(k, q; p_B) \simeq \frac{i}{(2\pi)^{-4}} \, \frac{\phi(k; p_B) \; \bar{\phi}(k; p_B)}{2\omega_B(p_0 - \omega_B + i\epsilon)}$$

- $\omega_B = \sqrt{M_B^2 + |\mathbf{p}|^2}$
- $\phi(k; p_B) \; \equiv$ Bethe-Salpeter Amplitude, in momentum space
- In configuration space,
  Bethe-Salpeter Amplitude $\rightarrow \langle 0 | T\{\phi_1(x_1)\phi_2(x_2)\} | p_B \, \beta \rangle$
  $\beta \equiv$ further quantum numbers

Close to the bound-state pole $p_0 \to \omega_B$

$$G \simeq G_B + \ regular\ \ terms$$

$\Rightarrow$ BS Equation

The integral equation determining the BS amplitude for a bound sys.

$$\phi(k; p_B, \beta) = G_0(k; p_B, \beta) \int d^4q' \, \mathcal{I}(k, q'; p_B) \, \phi(q'; p_B, \beta)$$

To simplify, nor self-energy neither vertex corrections, (at the present stage). For a two-scalar sys. the free-propagator is

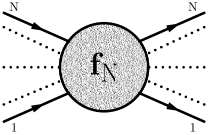
$$G_0 = \frac{i}{(\frac{p_B}{2} + k)^2 - m^2 + i\epsilon} \ \frac{i}{(\frac{p_B}{2} - k)^2 - m^2 + i\epsilon}$$

N.B. $\mathcal{I}(k, q'; p_B)$, the irreducible kernel in BSE, is the same one meets in

$$G = G + 0 + G_0 \, \mathcal{I} \, G$$

Feynman parametric integrals

In the sixties, Nakanishi (PR **130**, 1230 (1963)) proposed an integral representation of N-leg transition amplitudes, based on the parametric formula for the Feynman diagrams.



In a scalar theory, for N external legs, a generic contribution to the transition amplitude is given by

$$f_{\mathcal{G}}(p_1, p_2, ..., p_N) \propto \prod_{r=1}^{k} \int d^4 q_r \frac{1}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2) \ \ldots \ (\ell_n^2 - m_n^2)}$$

where one has n propagators and k loops ($\equiv$ number of integration variables).

The label $\mathcal{G} \to (n, k)$

Nakanishi Perturbative-theory Integral Rep.(PTIR) - I

Nakanishi proposal for a compact and elegant expression of the full N-leg amplitude $f_N(s) = \sum_{\mathcal{G}} \; f_{\mathcal{G}}(s)$

Introducing the identity

$$1 \doteq \prod_h \int_0^1 dz_h \delta\left(z_h - \frac{\eta_h}{\beta}\right) \int_0^\infty d\gamma \; \delta\left(\gamma - \sum_l \frac{\alpha_l m_l^2}{\beta}\right)$$

with $\beta = \sum \eta_i(\vec{\alpha})$ and integrating by parts $n - 2k - 1$ times

$$f_{\mathcal{G}}(s) \propto \; \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h) \; \tilde{\phi}_{\mathcal{G}}(z, \gamma)}{(\gamma - \sum_h z_h s_h)}$$

$\tilde{\phi}_{\mathcal{G}}(z,\gamma) \equiv$ proper function; $s \equiv \{s_h\}$ scalars from the ext. momenta
The dependence upon the details of the diagram, (n, k), moves from the denominator $\rightarrow$ the numerator!! The SAME formal expression for the denominator of ANY diagram $\mathcal{G}$ appears

Nakanishi PTIR - II

The full N-leg transition amplitude can be formally written as

$$f_N(s) = \sum_{\mathcal{G}} f_{\mathcal{G}}(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1-\sum_h z_h)\ \phi_N(z,\gamma)}{(\gamma - \sum_h z_h s_h)}$$

where

$$\phi_N(z,\gamma) = \sum_{\mathcal{G}} \tilde{\phi}_{\mathcal{G}}(z,\gamma)$$

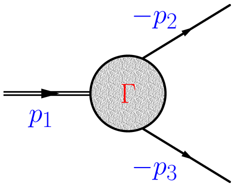
Within the BS framework, but using different kinematical variables, such an elegant expression can be exploited for obtaining

- the 3-leg transition amplitude (vertex function $\rightarrow$ bound-state BS amplitude) (Kusaka et al, PRD **56** (1997), Carbonell-Karmanov EPJA **27** (2006) 1, FSV PRD **89** (2014) 016010)
- the 4-leg one (off-shell or half-off-shell T-matrix $\rightarrow$ scattering-state BS amplitude) (FSV, PRD **85** (2012) 036009)

Vertex function for a scalar theory (N.B. for fermions $\rightarrow$ spinor indexes) by NIR

$$f_3(s) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z,\gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk\cdot p - i\epsilon}$$

with $p = p_1 + p_2$ and $k = (p_1 - p_2)/2$



QUESTION:

Can (NIR) of the vertex function Γ, elaborated within perturbation theory , be used in a non perturbative realm, as the BS framework does ?

ANSWER:

- Feynman Diagram framework $\equiv$ Nakanishi PTIR

$$\Gamma = \sum \textit{All Feynman Graph} \quad \Rightarrow \quad NIR$$

- Following the Bethe-Salpeter original work (PR 84 (1951) 1232): Γ can be obtained by a inhomogeneous integral equation (a non perturbative tool) with a kernel obtained from an infinite subset (the irreducible diagrams) of the graphs to be taken into account by Nakanishi, and iterating.
- The answer is : YES. NIR and BSE, with an analytical kernel, represent the same Γ.

Projecting BSE onto the LF hyper-plane $x^+ = 0$

- NIR contains some *hidden* freedom, once the weight function is taken as an unknown quantity.
- It is tempting to extend NIR to a non perturbative regime, needed for actually describing a bound state. Look for a dynamical equation to be fulfilled by the vertex function: the Bethe-Salpeter equation !
- Big Value: assuming an expression á la Nakanishi for the BS amplitude, then its analytic structure is displayed in full
- Within the non explicitly covariant LF framework the *valence component* for a two-scalar sys. is got by integrating on k^- the BS amplitude)

$$\psi_{n=2}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}}\, \xi\, (1-\xi) \int \frac{dk^-}{2\pi} \overbrace{\Phi_b(k,p)}^{\text{BS Amplitude}} =$$

$$= \frac{1}{\sqrt{2}} \xi\, (1-\xi)\, \underbrace{\int_0^\infty d\gamma' \, \frac{g_b(\gamma', 1-2\xi; \kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi-1)^2 \frac{M^2}{4} - i\epsilon]^2}}_{\text{NIR}}$$

LF projection of thehomogeneous BSE

$$\Phi(k,p) = G_0(k,p) \int d^4k' \, \mathcal{K}_{BS}(k,k',p) \, \Phi(k',p)$$

$\Rightarrow$

$$\int_0^\infty d\gamma' \frac{g_b(\gamma',z;\kappa^2)}{[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2-i\epsilon]^2} =$$
$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' \; V_b^{LF}(\gamma,z;\gamma',z') g_b(\gamma',z';\kappa^2).$$

with $V_b^{LF}(\gamma,z;\gamma',z')$ determined by the irreducible kernel $\mathcal{I}(k,k',p)$!

Ladder approx. by Carbonell and Karmanov within the explicitly-covariant LF framework (EPJA 27 (2006) 1 (also x-ladder in EPJA 27 (2006) 11). FSV PRD 89 (2014) 016010, non explicitly covariant version.

Very good agreement for both eigenvalues (the coupling constants at given binding energies) and LF distributions.

Wide phenomenology: (i) Scattering lengths in FSV EPJC 75 (2015) 398, (ii) spectra of excited states and LF momentum distributions in Gutierrez et al PLB 759 (2016) 131.

Spin dof and BSE

A two-fermion system, interacting in ladder approx. through

- a massive scalar

$$\mathcal{K}_S = \frac{g^2}{[(k-k')^2 - \mu^2 + i\epsilon]}$$

- .a massive pseudoscalar

$$\mathcal{K}_{PS} = \; - \frac{g^2}{[(k-k')^2 - \mu^2 + i\epsilon]}$$

- a massless vector

$$\mathcal{K}_V^{\mu\nu} = \frac{g^2 \; g^{\mu\nu}}{[(k-k')^2 + i\epsilon]}$$

as in Carbonell & Karmanov EPJA 46 (2010) 387. N.B. a form factor $F(K-k')$ at each vertex

BSE for fermions

$$\Phi(k,p) = S(p/2+k) \int d^4k' \, F^2(k-k') i\mathcal{K}(k,k')\Gamma_1 \, \Phi(k',p) \, \bar{\Gamma}_2 \, S(k-p/2)$$

$$S(q) = i\frac{\not{q}+m}{q^2-m^2+i\epsilon} \quad , \qquad F(k-k') = \frac{(\mu^2-\Lambda^2)}{[(k-k')^2-\Lambda^2+i\epsilon]}$$

$$\Gamma_1 = \Gamma_2 = 1 \ (scalar), \ \gamma_5 \ (pseudo), \ \gamma^\mu \ (vector)$$

$$\Phi(k,p) = S_1 \ \phi_1(k,p) + S_2 \ \phi_2(k,p) + S_3 \ \phi_3(k,p) + S_4 \ \phi_4(k,p)$$

$\phi_i \equiv$ unknown scalar functions, with well-defined symmetry under the exchange $1 \to 2$, from the symmetry of both $\Phi(k,p)$ ans S_i.

NIR applied to ϕ_i !!

$Tr\{S_i \ S_j\} = \mathcal{N}_i \ \delta_{ij}$ with

$$S_1 = \gamma_5 \quad , \quad S_2 = \frac{\not{p}}{M}\ \gamma_5 \quad , \quad S_3 = \frac{k\cdot p}{M^3}\not{p}\ \gamma_5 - \frac{1}{M}\not{k}\gamma_5 \quad , S_4 = \frac{i}{M^2}\sigma^{\mu\nu}p_\mu k_\nu \ \gamma_5$$

LF projection ⇒ integral-equation system

★For each ϕ_i, apply NIR

$$\psi_i(\gamma, z) = \int \frac{dk^-}{2\pi}\ \phi_i(k,p) = -\frac{i}{M}\int_0^\infty d\gamma' \frac{g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2 - i\epsilon]^2}$$

- $\gamma \equiv |\mathbf{k}_\perp|^2 \in [0, \infty]$ and $z \equiv 2x - 1 \in [-1, 1]$
- $\kappa^2 = 4m^2 - M^2$ with $M = 2m - B$.($B \equiv$ binding energy).

★ ★ The coupled-equation system

$$\psi_i(\gamma, z) = g^2 \sum_j \int_{-1}^{1} dz' \int_0^\infty d\gamma'\ g_j(\gamma', z'; \kappa^2)\ \mathcal{L}_{ij}(\gamma, z, \gamma', z'; p)$$

- $g_j(\gamma', z'; \kappa^2)$ are Nakanishi weights, eigenvectors of the integral-equation system.
- For actual calculations, a suitable basis Laguerre(γ) $\times$ Gegenbauer(z).
- The kernel $\mathcal{L}_{ij}(\gamma, z, \gamma', z'; p)$ contains singular contributions produced by integrating on k^- the combination of the numerator of the fermionic propagators and the operators S_i in $\Phi(k, p)$.

The non explicitly covariant LF framework allows one, in a straightforward way, to single out the singular contributions to $\mathcal{L}_{ij}$.

$$\text{For two} - \text{fermion BSE}: \quad \mathcal{C}_j = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} (k^-)^j \, \mathcal{S}(k^-, v, z, z', \gamma, \gamma')$$

with $j = 1, 2, 3$ and $\mathcal{S}(k^-, v, z, z', \gamma, \gamma')$ explicitly calculable

N.B., in the worst case

$$\mathcal{S}(k^-, v, z, z', \gamma, \gamma') \sim \frac{1}{[k^-]^2} \qquad \text{for} \quad k^- \to \infty$$

Then, one cannot close the arc at the ∞ for carrying out the needed analytic integration, but has to deal with singular behaviour, i.e. $\delta(x)$

The severity of the singularities, i.e. the power j, does not depend upon the complexity of the kernel.

★ ★The general rule says:

look at the constituent propagators and the structure of the BS amplitude, only

In the 70's, Yan et al studied the field theory in the Infinite Momentum frame,

The singular k^--integration involved in the investigation was one of the issues to be faced with.

Yan discussed (PRD 7 (1973) 1780) a singular integral like

$$\mathcal{I}(\beta, y) = \int_{-\infty}^{\infty} \frac{dx}{\left[\beta x - y \mp i\epsilon\right]^2} = \pm \frac{2\pi i \ \delta(\beta)}{\left[-y \mp i\epsilon\right]}$$

★ In the fermionic BSE case, one can rigorously evaluate the singular integrals by applying the Yan result and some simple extension, leading to derivative of the delta-functions (recall that we are using a basis, infinitely derivable,)

Differently, in the explicit covariant LF framework, the singular behavior of the relevant integrals was pragmatically healed by introducing a suitable smoothing function (Carbonell & Karmanov EPJA **46**, 387 (2010)).

Numerical comparison: Scalar coupling

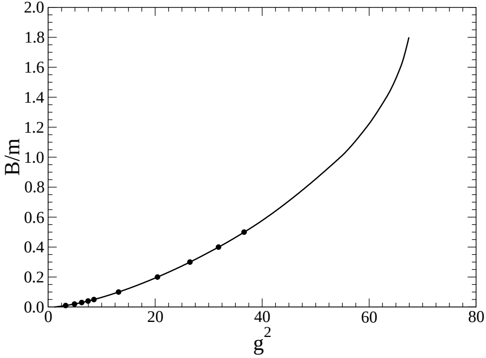
	$\mu/m = 0.15$			$\mu/m = 0.50$		
B/m	g^2_{dFSV}(full)	g^2_{CK}		g^2_{dFSV}(full)	g^2_{CK}	g^2_E
0.01	7.844	7.813		25.327	25.23	-
0.02	10.040	10.05		29.487	29.49	-
0.04	13.675	13.69		36.183	36.19	36.19
0.05	15.336	15.35		39.178	39.19	39.18
0.10	23.122	23.12		52.817	52.82	-
0.20	38.324	38.32		78.259	78.25	-
0.40	71.060	71.07		130.177	130.7	130.3
0.50	88.964	86.95		157.419	157.4	157.5
1.00	187.855	-		295.61	-	-
1.40	254.483	-		379.48	-	-
1.80	288.31	-		421.05	-	-

First column: binding energy.
Red digits: coupling constant g^2 for $\mu/m = 0.15$ and 0.50, with the analytical treatment of the fermionic singularities (present work). - Black digits: results for $\mu/m = 0.15$ and 0.50, with a numerical treatment of the singularities (Carbonell & Karmanov EPJA **46**, (2010) 387).
Blue digits: results in Euclidean space from Dorkin et al FBS. **42** (2008) 1.

Numerical comparison: Pseudo-Scalar coupling

	$\mu/m = 0.15$			$\mu/m = 0.50$	
B/m	g^2_{dFSV}(full)	g^2_{CK}		g^2_{dFSV} (full)	g^2_{CK}
0.01	225.7	224.8		422.6	422.3
0.02	233.2	232.9		430.5	430.1
0.04	243.1	243.1		440.9	440.4
0.05	247.1	247.0		444.9	444.3
0.10	262.1	262.1		460.4	459.9
0.20	282.9	282.9		482.1	480.7
0.40	311.7	311.8		513.3	515.2
0.50	322.9	323.1		525.8	525.9
1.00	362.3	-		570.9	-
1.40	380.1	-		591.8	-
1.80	388.7	-		602.1	-

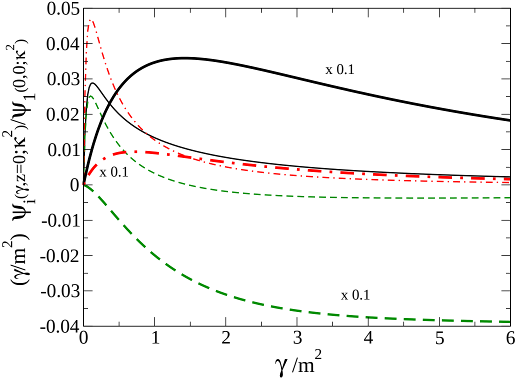
Numerical comparison: Vector coupling



Full dots: g^2 from Carbonell & Karmanov EPJA **46**, (2010) 387, with a numerical treatment of the singularities.

N.B. A critical value g_{crit} is clearly approached for $B/m \to 2$ (cf G. Baym PR 117 (1960) 886)

Vector coupling and high-momentum tails: $\gamma \equiv |\mathbf{k}_\perp|^2$



LF amplitudes ψ_i times γ/m^2 at fixed $z = 0$, for the vector coupling. $B/m = 0.1$ (thin lines) and 1.0 (thick lines).

—— : $(\gamma/m^2)\ \psi_1$.

— —: $(\gamma/m^2)\ \psi_2$.

— • : $(\gamma/m^2)\ \psi_4$.

$\psi_3 = 0$ for $z = 0$

Power one is expected for the pion valence amplitude from the dimensional arguments by X. Ji et al, PRL 90 (2003) 241601 (cf also Brodsky & Farrar for the counting rules of exclusive amplitudes)

For scalars $\phi(\gamma, z) \sim 1/[\gamma]^2$ (FSV PRD 89 (2014) 016010)

Conclusions & Perspectives

- A systematization of the technique for solving the fermionic BSE has been given, as well as a general rule for the expected singularities, that do not depend upon the complexity of the kernel.
- The LF framework has well-known advantages in performing analytical integrations, and in the investigated fermionic case its effectiveness has been shown in its full glory.
- Our numerical investigations, performed in ladder approximation at the present stage, confirm both the robustness of the Nakanishi Integral Representation for the BS amplitude, valid for any analytical BS kernel, and encourage to extended the technique to other interesting cases: boson-fermion system, and two interacting vectors.
- Calculations are in progress for the LF momentum distributions of the two-fermion system in the valence component, elucidating some formal subtleties.