Relativistic Yukawa model of Hamiltonian renormalization for bound states and scattering amplitudes

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Motivation

- Study simple problems to prepare for the difficult problems.
- The difficult problem: solving QCD for bound states on the light front using Renormalization Group Procedure for Effective Particles (RGPEP).
- The simple problem: an application of RGPEP in a simple model of field theory with Yukawa type of interactions.
- Simplifications: no small-x divergences, no gauge symmetry, simpler vertex, etc.
- The experience gained in the simple problem can be useful in understanding the running of effective quark-gluon coupling, which is needed in the precise calculations of bound states properties.

Outline

1. Model of Yukawa theory
2. Renormalization Group Procedure for Effective Particles (RGPEP)
3. Calculation of counterterms
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Canonical Hamiltonian

→ Starting point:

$$\mathcal{L}_f = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi \ ,$$

$$\mathcal{L}_I = -g\phi\bar{\psi}\psi \ .$$

→ Calculate Euler–Lagrange equations, solve constraints.

→ Density of canonical stress-energy tensor:

$$\begin{aligned} \mathcal{T}^{+-} &= -\frac{1}{2}(\partial^\perp \phi)^2 + \frac{1}{2}\mu^2\phi^2 \\ &+ \psi_+^\dagger (i\alpha^\perp\partial^\perp + \beta m + g\beta\phi)\frac{1}{i\partial^+}(i\alpha^\perp\partial^\perp + \beta m + g\beta\phi)\psi_+ \ , \end{aligned}$$

where $\psi_+ = \Lambda_+\psi$, $\Lambda_+ = \frac{1}{2}\gamma^0\gamma^+$.

→ The canonical Hamiltonian:

$$H_{\text{can}} = \int dx^- d^2x^\perp : \mathcal{T}^{+-} : \ .$$

Quantization I

→ Fourier decomposition of fields:

$$\hat{\psi}_+(x) = \Lambda_+ \sum_\sigma \int [p] \left[u_\sigma(p) b_{p\sigma} e^{-ipx} + v_\sigma(p) d^\dagger_{p\sigma} e^{ipx} \right] ,$$

$$\hat{\phi}(x) = \int [k] \left[a_k e^{-ikx} + a^\dagger_k e^{ikx} \right] ,$$

where,

$$\int [p] = \int_0^\infty \frac{dp^+}{4\pi p^+} \int \frac{d^2 p^\perp}{(2\pi)^2} .$$

Quantization II

→ Canonical (anti)commutation relations:

$$\begin{aligned} i[\phi(x^-,x^\perp),\phi(y^-,y^\perp)] &= \frac{1}{4}\epsilon(x^- - y^-)\delta^{(2)}(x^\perp - y^\perp) \ , \\ \{\psi_+(x^-,x^\perp),\psi_+^\dagger(y^-,y^\perp)\} &= \Lambda_+\delta(x^- - y^-)\delta^{(2)}(x^\perp - y^\perp) \ , \end{aligned}$$

or

$$\begin{aligned} [a_p, a_{p'}^\dagger] &= 2(2\pi)^3 p^+\delta(p^+ - p'^+)\delta^{(2)}(p^\perp - p'^\perp) \ , \\ \{b_{p\sigma}, b_{p'\sigma'}^\dagger\} &= 2(2\pi)^3 p^+\delta(p^+ - p'^+)\delta^{(2)}(p^\perp - p'^\perp)\delta_{\sigma\sigma'} \ , \\ \{d_{p\sigma}, d_{p'\sigma'}^\dagger\} &= 2(2\pi)^3 p^+\delta(p^+ - p'^+)\delta^{(2)}(p^\perp - p'^\perp)\delta_{\sigma\sigma'} \ . \end{aligned}$$

Fock space truncation I

To simplify we throw out all of the Fock space sectors except the sector with one fermion and a sector with one fermion and one boson. Notation:

$$
\begin{aligned}
&|0\rangle && \leftarrow \text{vacuum}, \\
&|1\rangle = b^\dagger_{p_1\sigma_1}|0\rangle && \leftarrow \text{one fermion state}, \\
&|2\rangle = b^\dagger_{p_2\sigma_2} a^\dagger_{k_2}|0\rangle && \leftarrow \text{one fermion} + \text{one boson state}.
\end{aligned}
$$

Fock space truncation II

Candidate for Model Hamiltonian:

$$PH_{\text{can}}P = \int_1 p_1^- |1\rangle\langle 1| + \int_2 \left(p_2^- + k_2^-\right) |2\rangle\langle 2|$$

$$+ g \int_{21} \tilde{\delta}^{(3)}(2-1)\ \bar{u}_{\sigma_2}(p_2) u_{\sigma_1}(p_1) |2\rangle\langle 1| + H.c.$$

$$+ g^2 \int_{22'} \tilde{\delta}^{(3)}(2-2')\ \bar{u}_{\sigma_2}(p_2) \frac{\gamma^+}{2\mathcal{P}^+} u_{\sigma_2'}(p_2') |2\rangle\langle 2'| \ ,$$

where

$$P = \int_1 |1\rangle\langle 1| + \int_2 |2\rangle\langle 2| \ ,$$

$$\int_1 = \sum_{\sigma_1} \int [p_1] \ , \quad \int_2 = \sum_{\sigma_2} \int [p_2 k_2] \ .$$

Regularization

Model Hamiltonian reads:

$$H_{\text{model}} = \int_1 p_1^- |1\rangle\langle 1| + \int_2 \left(p_2^- + k_2^-\right) |2\rangle\langle 2|$$

$$+ g \int_{21} \theta_2^\Lambda \; \tilde{\delta}^{(3)}(2-1) \; \bar{u}_{\sigma_2}(p_2) u_{\sigma_1}(p_1) |2\rangle\langle 1| + H.c.$$

$$+ g^2 \int_{22'} \theta_2^\Lambda \theta_{2'}^\Lambda \; \tilde{\delta}^{(3)}(2-2') \; \bar{u}_{\sigma_2}(p_2) \frac{\gamma^+}{2\mathcal{P}^+} u_{\sigma_2'}(p_2') |2\rangle\langle 2'|$$

$$+ X_\Lambda \,,$$

where

$$\theta_2^\Lambda = \theta(\Lambda^2 - \mathcal{M}^2(x,\kappa))$$

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Renormalization Group Procedure for Effective Particles I

The key idea of RGPEP is to switch from bare particles – point particles – to effective ones with nonzero size by unitary rotation of the annihilation and creation operators,

$$q_t = \mathcal{U}_t \, q_0 \, \mathcal{U}_t^\dagger \ .$$

The rotation is such that effective particles do not couple unless the difference in free invariant mass between ingoing and outgoing states is smaller than $\lambda = t^{-1/4}$.
The Hamiltonian of the theory can be written in either base

$$H_t(q_t) = H_0(q_0) \ .$$

Renormalization Group Procedure for Effective Particles II

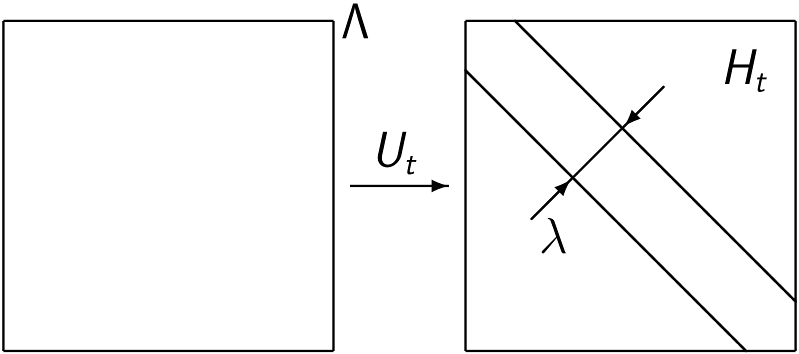

Figure: Initial theory → effective theory. RGPEP introduces form factors into vertices, which narrow the Hamiltonian in free-states basis.

Renormalization Group Procedure for Effective Particles III

The narrowing of the Hamiltonian is realized by the following differential equation,

$$\frac{d}{dt}\mathcal{H}_t = [[\mathcal{H}_f, \mathcal{H}_{Pt}], \mathcal{H}_t] \ ,$$

where (in the model)

$$\begin{aligned}
\mathcal{H}_f &= \int_1 p_1^- |1\rangle\langle 1| + \int_2 \left(p_2^- + k_2^-\right) |2\rangle\langle 2| \ , \\
\mathcal{H}_t &= \mathcal{H}_f + \int_{21} \mathcal{H}_t(2;1)\ |2\rangle\langle 1| + H.c. \\
&+ \int_{22'} \mathcal{H}_t(2;2')\ |2\rangle\langle 2'| + \int_{11'} \mathcal{H}_t(1;1')|1\rangle\langle 1'| \ , \\
\mathcal{H}_{Pt} &= \int_{21} (\mathcal{P}^+)^2 \mathcal{H}_t(2;1)\ |2\rangle\langle 1| + H.c. \\
&+ \int_{22'} (\mathcal{P}^+)^2 \mathcal{H}_t(2;2')\ |2\rangle\langle 2'| + \int_{11'} (\mathcal{P}^+)^2 \mathcal{H}_t(1;1')|1\rangle\langle 1'| \ .
\end{aligned}$$

Renormalization Group Procedure for Effective Particles IV

The initial condition is H_0 and in the model

$$H_0 = H_{\rm model} \ .$$

In the lowest order the $\mathcal{H}_t$ in the model is

$$\mathcal{H}_t = \int_1 p_1^- |1\rangle\langle 1| + \int_2 \left(p_2^- + k_2^-\right) |2\rangle\langle 2|$$

$$+ g \int_{21} \mathcal{H}_{t2}(2;1) \ |2\rangle\langle 1| + H.c. \qquad + g^2 \int_{22'} \mathcal{H}_{t2}(2;2') \ |2\rangle\langle 2'|$$

$$+ g^2 \int_1 \frac{\widetilde{\mathcal{H}}_{t2}(1;1)}{p_1^+} |1\rangle\langle 1| + O(g^3) \ ,$$

$$\mathcal{H}_{t2}(2;1) = \theta_2^\Lambda \ \tilde{\delta}^{(3)}(2-1) \ e^{-t(\mathcal{M}_2^2 - m^2)^2} \bar{u}_{\sigma_2}(p_2) u_{\sigma_1}(p_1) \ ,$$

$$\mathcal{H}_{t2}(2;2') = \theta_2^\Lambda \theta_{2'}^\Lambda \ \tilde{\delta}^{(3)}(2-2') \ e^{-t(\mathcal{M}_2^2 - \mathcal{M}_{2'}^2)^2} \widetilde{\mathcal{H}}_{t2}(2;2') \ .$$

Prescription for the counterterms

1. Calculate the effective theory.
2. If any matrix element of the effective theory is divergent when $\Lambda \to \infty$, then add appropriate counterterm to the initial theory (unique up to finite part).
3. Check **every** matrix element.
4. Constrain finite parts of counterterms by symmetry requirements and experimental data.
5. In perturbation theory, repeat steps 1–4 order by order.

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Calculation of the counterterms in the model I

2^{nd}order mass counterterm

→ Effective mass term in the 2^{nd}order:

$$\begin{aligned}\widetilde{\mathcal{H}}_{t2}(1;1) &= \widetilde{\mathcal{H}}_{02}(1;1) \\ &+ \int [x\kappa]\theta_2^{\Lambda} \frac{e^{-2t(\mathcal{M}^2-m^2)^2}-1}{\mathcal{M}^2(x,\kappa)-m^2} \\ &\quad \times \sum_{\sigma_2} \bar{u}_{\sigma_1}(p_1)u_{\sigma_2}(p_2)\bar{u}_{\sigma_2}(p_2)u_{\sigma_1}(p_1)\ .\end{aligned}$$

→ Divergent part:

$$\int [x\kappa]\theta_2^{\Lambda} \frac{-1}{\mathcal{M}^2-m^2} \sum_{\sigma_2} \bar{u}_{\sigma_1}(p_1)u_{\sigma_2}(p_2)\bar{u}_{\sigma_2}(p_2)u_{\sigma_1}(p_1) \approx$$

$$-\frac{1}{16\pi^2}\left[\frac{1}{2}\Lambda^2 - \mu^2 \log\frac{\Lambda^2}{\mu^2} + 3m^2 \log\frac{\Lambda^2}{m^2} + finite\right]\ .$$

Calculation of the counterterms in the model II

2^{nd}order mass counterterm

Therefore, the counterterm needs to be of the form

$$\widetilde{\mathcal{H}}_{02}(1;1) = \int [x\kappa]\theta_2^\Lambda \frac{1}{\mathcal{M}^2 - m^2} \sum_{\sigma_2} \bar{u}_{\sigma_1}(p_1) u_{\sigma_2}(p_2) \bar{u}_{\sigma_2}(p_2) u_{\sigma_1}(p_1) + \widetilde{\mathcal{H}}_{02}^{\text{finite}} .$$

Useful representation:

$$\widetilde{\mathcal{H}}_{02}(1;1) = \omega^2 + 2m^2 \left[\alpha(m^2) + \beta(m^2)\right] + \widetilde{\mathcal{H}}_{02}^{\text{finite}} .$$

Calculation of the counterterms in the model I

Finite part of the 2^{nd} order mass counterterm

To find finite part of the mass counterterm we consider an eigenproblem,

$$H_t |\mathcal{P}\sigma\rangle_{phys,t} = \frac{m_{phys}^2 + \mathcal{P}^{\perp 2}}{\mathcal{P}^+} |\mathcal{P}\sigma\rangle_{phys,t} \ ,$$

where

$$\begin{aligned} |\mathcal{P}\sigma\rangle_{phys,t} &= \int_1 \mathcal{P}^+ \tilde{\delta}^{(3)}(\mathcal{P} - p_1) c^{\sigma}_{\sigma_1 t} |1\rangle_t \\ &+ \int_2 \mathcal{P}^+ \tilde{\delta}^{(3)} \left(\mathcal{P} - p_2 - k_2\right) \phi^{\sigma}_{\sigma_2 t}(x, \kappa) |2\rangle_t \ , \end{aligned}$$

and solve it up to 2^{nd} order in the expansion in the coupling constant.

Calculation of the counterterms in the model II

Finite part of the 2^{nd}order mass counterterm

Solution exists if

$$m = m_{phys} \ ,$$

and the finite part of the mass counterterm is fixed to

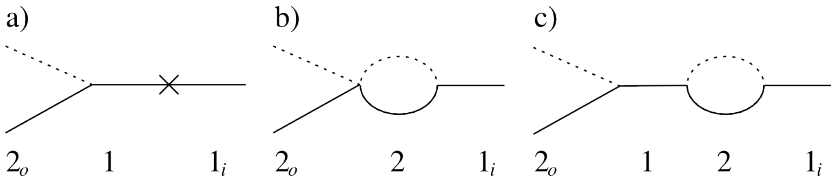
$$\widetilde{\mathcal{H}}_{02}^{\text{finite}} = 0 \ .$$

Effective wavefunction of the fermion

$$\phi^{\sigma}_{\sigma_2 t}(x_2, \kappa_2) = g\,\frac{e^{-t(\mathcal{M}_2^2 - m^2)^2}}{m^2 - \mathcal{M}_2^2} \sum_{\sigma'} \bar{u}_{\sigma_2}(p_2) u_{\sigma'}(\mathcal{P}) c^{\sigma}_{\sigma' t} + O(g^3)$$

Calculation of the counterterms in the model I

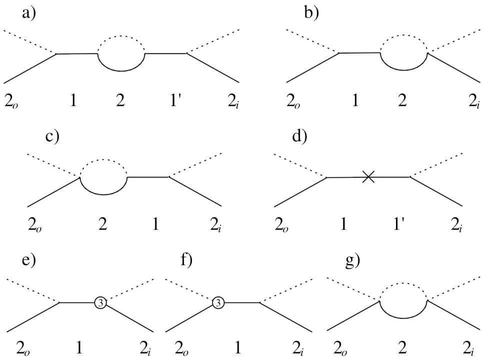
3rd order interaction counterterm



$$\widetilde{\mathcal{H}}_{03}(2_o;1_i) = m\left[\alpha(m^2)+\beta(m^2)+A\right]\bar{u}_{\sigma_{2o}}(p_{2o})\frac{\gamma^+}{2\mathcal{P}^+}u_{\sigma_{1i}}(p_{1i}) + \frac{\alpha(m^2)+B}{2}\bar{u}_{\sigma_{2o}}(p_{2o})u_{\sigma_{1i}}(p_{1i}) \; .$$

Calculation of the counterterms in the model I

4^{th} order interaction counterterm



$$\widetilde{\mathcal{G}}_{04}(2_o;2_i) = \left[\alpha(m^2) + C\right] \bar{u}_{\sigma_{2o}}(p_{2o}) \frac{\gamma^+}{2\mathcal{P}^+} u_{\sigma_{2i}}(p_{2i}) \ .$$

Scattering matrix

$$\widetilde{T}_4 = \bar{u}_{\sigma_{2o}}(p_{2o}) \left[\Gamma_1(\mathcal{P}^2)\not{\mathcal{P}} + \Gamma_2(\mathcal{P}^2) m + \Gamma_3(\mathcal{P}^2) \frac{\gamma^+}{2\mathcal{P}^+} \right] u_{\sigma_{2i}}(p_{2i}) \,, \quad (1)$$

where Γ_1, Γ_2 and Γ_3 are finite and

$$\Gamma_3(\mathcal{P}^2) \quad = \quad B - C - \frac{2m^2}{\mathcal{S}_i} A \,. \qquad (2)$$

Lorentz invariance

$$\Gamma_3 = 0 \,. \qquad (3)$$

Therefore,

$$A = 0 \,, \qquad B = C \,. \qquad (4)$$

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Effective 3rd order vertex I

$$\mathcal{H}_t(2;1) = \theta_2^\Lambda \,\tilde{\delta}(2-1)\; e^{-t(\mathcal{M}_2^2 - m^2)^2} \times \tilde{g}_t(\mathcal{M}_2^2) \,\cdot\, \bar{u}_{\sigma_2}(p_2) \left[1 + \delta m_t(\mathcal{M}_2^2) \frac{\gamma^+}{2\mathcal{P}^+} \right] u_{\sigma_1}(p_1)\,,$$

$$\tilde{g}_t(\mathcal{M}_2^2) \equiv g + B_t(\mathcal{M}_2^2) g^3$$

$$g_t = \tilde{g}_t(m^2)\,. \qquad (5)$$

$$B_t(m^2) \overset{m^4 t \ll 1}{=} -\frac{1}{128\pi^2}\left[\log\left(2m^4 t\right) + \gamma - 3 + O\left(\sqrt{m^4 t}\right)\right] - \frac{B}{2}\,,$$

Effective 3rd order vertex II

$$g_t = g_{t_0} + \frac{g_{t_0}^3}{32\pi^2} \log \frac{\lambda}{\lambda_0} , \qquad \lambda \gg m ,$$

where $\lambda = t^{-1/4}$ and $\lambda_0 = t_0^{-1/4}$.

$$g_\Lambda = g_{\Lambda_0} + \frac{g_{\Lambda_0}^3}{32\pi^2} \log \frac{\Lambda}{\Lambda_0} , \qquad \Lambda \gg m ,$$

$$g_\Lambda = g_{\mathcal{M}_0} + \frac{g_{\mathcal{M}_0}^3}{32\pi^2} \log \frac{\mathcal{M}}{\mathcal{M}_0} , \qquad \mathcal{M} \gg m .$$

Conlcusions

Our simple calculation:

- is an illustrative example of application of RGPEP (in the context of dealing with ultraviolet divergences);
- helps in understanding the concept of running effective coupling constant in Minkowski spacetime;
- due to presence of similar structures, it can be helpful when calculating effective quark-gluon coupling constant in QCD.

Thank you.